Unit Systems, E&M Units, and Natural Units

Unit Soup

We live in a soup of units—especially in the U.S. where we are saddled with imperial units that even the former empire has now dropped. I was once—like many scientists and most international students in the U.S. are today—vocally irritated and perplexed by this fact. Why would the U.S.—who likes to claim #1 status in all things even when not true—be behind the global curve here, when others have transitioned? I only started to understand when I learned machining practices in grad school. The U.S. has an immense infrastructure for manufacturing, comprised largely of durable machines that last a lifetime. The machines and tools represent a substantial capital investment not easily brushed aside. It is therefore comparatively harder for a (former?) manufacturing powerhouse to make the change. So we become adept in two systems. As new machines often build in dual capability, we may yet see a transformation decades away.

But even leaving aside imperial units, scientists bicker over which is the "best" metric-flavored unit system. This tends to break up a bit by generation and by field. Younger scientists tend to have been trained in SI units, also known as MKSA for Meters, Kilograms, Seconds, and Amperes. Force, energy, and power become Newtons, Joules, and Watts. Longer-toothed scientists often prefer c.g.s. (centimeter, gram, seconds) in conjunction with Gaussian units for electromagnetic problems. Force, energy, and power become dynes, ergs, and ergs/second. Yuck. I'd like a one-billion erg-per-second light bulb, please? The MKSA system meshes nicely with the human scale. Then you have "those people" who don't often measure things (theorists) with a penchant for "natural units" in which $c = h = k_B = G = 1$. We'll get to this and exploit some of its usefulness at the end.

SI Units

In this course we will concentrate on SI (MKSA) units, as it is likely the most familiar and consistent with other courses. Everything goes smoothly as long as we stick to the MKS part. It's in electromagnetism that things go off the rails a bit. But before deciding that Gaussian units are far better, see the Wikipedia page (`https://en.wikipedia.org/wiki/Gaussian_units`) comparing the two. Things like Coulomb's law look cleaner in Gaussian units, but Maxwell's equations (and even the Lorentz force) are cleaner in MKSA: fewer factors of $1/c$ and 4π thrown in. But yes, there are pros and cons to each and no "right" answer.

Because the standard combinations of M, K, and S are straightforward and likely need no elaboration, we will jump straight to the morass of units in the MKSA/SI system

SI Units for E&M

Charges are in Coulombs, with an electron having a charge 1.602×10^{-19} C. Is it a coincidence that this is numerically the same as the number of Joules in an electron-volt? Of course not! We'll come back to this. Let's start with the force law:
$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2},$$
which we know to be Newtons, or kg·m/s^2. We know what everything is except ε_0, so we can deduce that it must have units of C^2/N · m^2, or C^{2}s^2/kg · m^3.

To prevent confusion, the in-line notation here will only have one / sign, so everything to the right is in the denominator. Note that a **summary table** appears below, which may be useful to refer to as you go along through what follows.

We also know that the electric potential is rendered in volts (V), and is defined so that an energy is some quantity like $U \sim qV$ (akin to the force being $F \sim qE$). So we have that Volts must look dimensionally like

Table 1: Summary table of E&M SI units

quantity	Unit	SI	Other
charge	C	C	—
current	A	C/s	—
elec. potential	V	$\frac{\text{kg} \cdot \text{m}^2}{\text{C} \cdot \text{s}^2}$	J/C
electric field	—	$\frac{\text{kg} \cdot \text{m}}{\text{C} \cdot \text{s}^2}$	V/m, N/C
permittivity (ε_0)	—	$\frac{\text{C}^2 \text{s}^2}{\text{kg} \cdot \text{m}^3}$	F/m
permeability (μ_0)	—	$\frac{\text{kg} \cdot \text{m}}{\text{C}^2}$	H/m
resistance	Ω	$\frac{\text{kg} \cdot \text{m}^2}{\text{C}^2 \text{s}}$	V/A
capacitance	F	$\frac{\text{C}^2 \text{s}^2}{\text{kg} \cdot \text{m}^2}$	C/V, s/Ω
inductance	H	$\frac{\text{kg} \cdot \text{m}^2}{\text{C}^2}$	$\Omega \cdot$ s
magnetic field	T	$\frac{\text{kg}}{\text{C} \cdot \text{s}}$	10^4 gauss

$q/\varepsilon_0 r$, and electric field must look like $q/\varepsilon_0 r^2$. Therefore V = kg $\cdot$ m^2/C $\cdot$ s^2 (also J/C) and the electric field is in V/m = kg $\cdot$ m/C $\cdot$ s^2 (also N/C).

Let's verify that things are on track: power is current times voltage: $P = IV$. Current is in A≡C/s, so power would come out to kg $\cdot$ m^2/s^3 = J/s = W. So far so good. Let's also revisit the electron volt, which should be the energy attained by an electron crossing one volt of potential. 1.602×10^{-19} C times 1 V, or 1 kg $\cdot$ m^2/C $\cdot$ s^2 is 1.602×10^{-19} kg $\cdot$ m^2/s^2, or 1.602×10^{-19} J, as expected.

Let's now visit practical electrical (or even electronics) relationships. Ohm's Law says $V = IR$, so resistance must have units: Ω = kg $\cdot$ m^2/C^2 $\cdot$ s. We can get capacitance two ways. One is knowing that $C \sim Q/V$, the other is knowing that the complex impedance of a capacitor is $Z = \frac{1}{i\omega C}$, in Ohms. Either way, we get that a Farad is F = C^{2}s^2/kg $\cdot$ m^2. Likewise for inductance, either use $V = L\frac{dI}{dt}$, or impedance $Z = i\omega L$, so a Henry has units H = kg $\cdot$ m^2/C^2. It is also easy to verify now that energy will go like $I^2 L$ or CV^2 (with a factor of $\frac{1}{2}$ in both cases).

Returning to permittivity, ε_0 had units C^{2}s^2/kg $\cdot$ m^3, which we can now recognize as F/m. In fact, $\varepsilon_0 \approx$ 9 pF/m (8.85×10^{-12}). A parallel-plate capacitor ought to have capacitance proportional to the area and inversely proportional to separation of the plates. For an air/vacuum gap, we have $C \sim \varepsilon_0 A/s$. A circuit board with $A = 0.01$ m^2 and thickness 0.001 m will have a natural capacitance of 90 pF. But actually more by a factor of about 4.5 due to the dielectric, or about 400 pF.

Since we know $c^2 = (\mu_0 \varepsilon_0)^{-1}$, we get that magnetic permeability, μ_0, must have units of kg $\cdot$ m/C^2, which is also H/m (akin to ε_0 as F/m). The numeric value is $4\pi \times 10^{-7}$ H/m. The inductance of a copper trace on a circuit board is $\mu_0 \ell/2\pi$ times the natural log of the trace aspect ratio (approximately) as a small numerical addendum. For typical geometries, we may be talking $\sim$ 5–10 nH per cm (or about 5×10^{-7} H/m; very close to μ_0).

We can now do cool things with coaxial cables. If we know the capacitance per unit length (same units as ε_0) and the inductance per unit length (sam units as μ_0), we can say that the cable impedance must go like $\sqrt{L'/C'}$, and that the transmission velocity will go like $v = 1/\sqrt{L'C'}$. The inductance and capacitance per unit length are set by geometry, and the dielectric constant between the central conductor and the outer shell. Typical values are 100 pF/m and 0.25 μH/m, leading to a 50 Ω impedance and a transmission velocity of $v = 0.66c$.

When dealing with polarization of materials, the electric displacement looks like $D = \varepsilon E$, so has units of $\varepsilon_0 E$, which we can now put together to get C/m^2. This is another way of expressing a space density of dipoles, each carrying units C $\cdot$ m.

Magnetic fields have units of Tesla, and we can get at this via the Bio-Savart law: $B = \mu_0 I/2\pi r$. We got the units for μ_0 above, and can therefore say that a Tesla is kg/C·s. Note that the units for electric field, kg·m/C·s^2 are Tesla times velocity. Thus the relationship in an electromagnetic wave that $B_0 = \frac{1}{c}E_0$ (remember by the fact that the magnetic field is much smaller than the electric field, numerically).

We can now check/verify that we get the right formulas for energy in electric and magnetic fields. You might ask: based on the vague memory that energy goes like E^2 and like B^2, what do I have to multiply by to get Joules? Well, E^2 is $kg^2 \cdot m^2/C^2 \cdot s^4$, so to get $kg \cdot m^2/s^2$, I need to multiply by $C^2 \cdot s^2/kg$, which is just like $\varepsilon_0 V$ (permittivity times a volume). So $\varepsilon_0 E^2$ is an energy density (or pressure, it turns out). A similar exercise with the magnetic field results in an equivalent energy density via B^2/μ_0. An electromagnetic wave, with $B_0 = \frac{1}{c}E_0$ therefore has an energy density $\varepsilon_0 E_0^2 + B_0^2/\mu_0 = \varepsilon_0 E_0^2 + E_0^2/c^2\mu_0 = \varepsilon_0 E_0^2 + \varepsilon_0 E_0^2$, so that both electric and magnetic fields carry the same energy density.

Estimating Pull of a Magnet

Okay, we slogged our way through some pretty dry stuff to build a table of units we have little hope of remembering. Let's work on some rewards. For all your exposure to E&M courses, if you were asked to estimate how much mass a strong (Nd, for instance) magnet could lift, do you have the tools? The answer is yes, even if the classes did not themselves make it clear.

We start by asking what generates magnetism in materials. Circular charge currents within atoms are to blame. This could be in the form of intrinsic spin (electrons, nucleus), and/or in the form of orbital angular momentum. And, of course, one must have coherent net alignment among the atoms (ferromagnetism) or it all washes out.

Let's assume that each atom carries $\hbar$ angular momentum in some form, and that all atoms contribute coherently. Planck's constant has units J·s, which unpacks to $kg \cdot m^2/s$. Note that the familiar construction for angular momentum in mechanics contexts is $I\omega$, having units $(kg \cdot m^2)(s^{-1})$, which is reassuringly the same.

For conceptual simplicity, we model the orbital angular momentum of an electron in an atom as corresponding to a radius of $r = 10^{-10}$ m (1 Å), having mass $m = 9 \times 10^{-31}$ kg. The moment of inertia is $I = mr^2$, and angular velocity is $\omega = v/r$, for some orbital velocity, v (note we are treating the electron classically, which is fine for establishing the numerics). We set the angular momentum to have magnitude $\hbar$, so that $mr^2\frac{v}{r} = mrv = \hbar$, so $v = \hbar/mr$ (which computes to 10^6 m/s, or $c/300$, incidentally).

The current associated with sending $e = 1.602 \times 10^{-19}$ C around circumference $2\pi r$ at this velocity is $i = e/T = ev/2\pi r = e\hbar/2\pi mr^2$, after inserting our relation for v. Applying the Biot-Savart law, the magnetic field is

$$B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 e\hbar}{2m(2\pi^2 r^3)} = \mu_B \frac{\mu_0}{2\pi r^3},$$

where we have identified and pulled out the Bohr magneton, μ_B, which has units $C \cdot m^2/s$, looking like current times area, or alternatively, J/T. Note that the denominator is a volume, so $\mu_B \mu_0$ is a magnetic field density. Shoving in numbers, we estimate:

$$\frac{4\pi \times 10^{-7} \cdot 1.6 \times 10^{-19} \cdot 10^{-34}}{4\pi \cdot \pi \cdot 10^{-30} \cdot 10^{-30}} \approx \frac{1}{2}\frac{10^{-60}}{10^{-60}} = \frac{1}{2}\ \text{T}.$$

We have taken an approximate (5% accurate) round-number for $\hbar$, and a 10% accurate rounding of the electron mass.

So half a Tesla, huh? The Wikipedia page on the Tesla unit puts the field at 1.25 T at the surface of a neodymium magnet. So we're nicely in a useful ballpark!

But how much mass can the magnet lift?

Let's take the magnet to be something like a cubic centimeter. The magnetic field emerging from the contact surface will have an associated area of about 1 cm^2. The key is to evaluate the energy in the field near the magnet. Once a piece of metal is attached, the field lines "short circuit" through the metal and so the field that was once present is no longer there, so its energy is gone. What we then care about is how much field energy exists within, say, 1 mm of the magnet, which by comparing to gravitational potential energy of an object lifted through that millimeter will tell us how much mass might be lifted through that distance.

The energy in the magnetic field is approximately

$$W = \frac{B^2}{\mu_0} V = \frac{B^2}{\mu_0} \Delta h \cdot A \approx \frac{(1\,\mathrm{T})^2}{4\pi \times 10^{-7}} (10^{-3}\,\mathrm{m})(10^{-4}\,\mathrm{m}^2) \sim 0.1\,\mathrm{J}.$$

Equating this with a potential energy change for the lifted object, $mg\Delta h$, we compute that the associated mass is about 10 kg. Totally believable that a strong Nd magnet can pull that much mass through the final millimeter, if you've ever played with one. Note that to the extent the magnetic field is constant, the force saturates. In other words, if the final millimeter has the same ~ 1 T field throughout, we would compute the same mass for a lift (Δh) of half-a millimeter, for instance. As we get farther from the magnet, we expect a rapid ($1/r^3$) decline in magnetic field strength, but in the very near field ($\Delta h \ll R$, where R is the magnet radius) we may expect a relatively flat field profile.

Natural Units

Finally, for the non-measuring types among us (a.k.a., theorists), we ought to cover the concept of natural units. Here, we set the fundamental constants of nature to unity, so that we jettison numerical baggage. This looks something like:

$$h = c = k_B = G = 1.$$

In doing so, we make the following unit associations:

$$[Energy] = [Mass] = [Temperature] = [Length]^{-1} = [Time]^{-1}. \tag{1}$$

This can be more directly seen by the relations:

$$E = mc^2 = k_B T = \frac{hc}{\lambda} = h\nu$$

in a parallel construction to Eq. 1. Just set each of the physical constants to unity (with no units) and the heuristic form in Eq. 1 pops out. For instance, $E = mc^2$ reduces simply to $E = m$ (first relation in Eq. 1) when we set $c = 1$. Moreover, by making c dimensionless, we make length and time the same unit (last relation in Eq. 1). In this system of "units," c is so fundamental that length and time are intimately connected by light travel. When we use light years, we're effectively doing just this. In this way of thinking, length is time and vice versa.

Also elucidating relationships a bit: $\Delta E \Delta t \sim h$, and Planck's constant is unity and dimensionless, so that we see the inverse relationship between energy and time, as indicated in Eq. 1. Likewise, hc would have units of $[Energy] \cdot [Length]$ in the normal system of units, but when we define each constant as 1, we set energy and length as inverses of the other. But this is saying nothing new. Once we've equated length and time through $c = 1$, the inverse between energy and time at the beginning of the paragraph implies a similar inverse between energy and length.

The combination hc (or barred equivalent) is especially useful when working in natural units as a conversion between energy and length. The idea is to work in natural units and then throw in appropriate powers of c, hc, etc. until things make sense. In SI units, $hc = 1.986 \times 10^{-25}\,\mathrm{J} \cdot \mathrm{m}$, but this might be more conveniently expressed in various flavors of eV, such as

$$hc = 1.24 \times 10^{-6}\,\mathrm{eV} \cdot \mathrm{m} = 1.24\,\mathrm{eV} \cdot \mu\mathrm{m} = 1.24\,\mathrm{keV} \cdot \mathrm{nm} = 1.24\,\mathrm{GeV} \cdot \mathrm{fm},$$

etc. Or one might prefer the barred version: $0.197\,\mathrm{eV} \cdot \mu\mathrm{m}$ and so on. Depending on the system of interest, (electrons in devices or nucleons), the choice of scale for energy and length change. But right away we see that 1 eV phenomena have an associated (wave)length around a micron, and that nuclear conditions happen at the fermi level (10^{-15} m). The LHC, probing TeV energies, is working at the 10^{-18} m length scale.

For thermal phenomena, setting $k_B = 1$ (otherwise 1.38×10^{-23} J/K) translates to 8.6×10^{-5} eV/K, or its inverse 11,600 K/eV. So right away we have a sense for relevant temperature scales. At 300 K, we're looking at something like 0.025 eV, or 1/40 eV.

For masses, you are probably already familiar with energy equivalents to mass, so that electrons are 511 keV and protons are 938 MeV, etc.

As an example of how to work in natural units, once we discard constants, the Bohr radius simplifies to the fundamental (physical) constituents. By this, we mean that

$$a_0 = \frac{\hbar^2}{m_e e^2} \rightarrow \frac{1}{m_e e^2}.$$

In other words, forget the numerical constant(s) and just think of the Bohr radius as an inverse mass (as per Eq. 1). It also depends on the inverse of electric field strength, as embodied in e^2. But what are the units of e^2? In natural units, this is a dimensionless measure of field strength relative to fundamental constants. We know this as the fine structure constant: $\alpha = e^2/\hbar c \approx 1/137$. In other words, electromagnetism is about two orders-of-magnitude weaker than the "natural" level. The smallness of this number allows perturbative treatments in ways that the strong force, with a much larger coupling, does not. In order to turn $1/me^2$ into a length with actual units, we turn the mass into energy via mc^2, then into length via $\hbar c$, and likewise turn e^2 into something meaningful via $\hbar c$. Thus we have:

$$a_0 = \frac{1}{m_e e^2} = \frac{\hbar c}{m_e c^2} \frac{\hbar c}{e^2} \approx \frac{1}{2\pi} \frac{1.24\,\text{keV} \cdot \text{nm}}{511\,\text{keV}} \frac{1}{\alpha} \approx \frac{137}{6} \frac{1\,\text{nm}}{400} \approx \frac{1}{18}\,\text{nm} \approx 0.5 \times 10^{-10}\,\text{m},$$

where we used one of the (convenient) values for hc above and divided by 2π to deal with the bar in $\hbar$.

For more elaborate unit combinations, like number density or mass/energy density, we can refer to Eq. 1 to understand how to proceed. A number density goes like $[Length]^{-3}$, which means it goes like $[Energy]^3$. A pressure (also an energy density) therefore goes like $[Energy]^4$.

In a system of natural units, one might ask what "unit mass" or "unit length" are. They surely will not any longer be kilograms and meters! The answer, generically, is Planck Units. These are various combinations of the fundamental constants that yield dimensions of length, mass, etc. For example, in SI units, G has m, kg, and s in some arrangement, c has m and s, and $\hbar$, like G, has kg, m, and s in a different combination. Combining these three in different ways can isolate just meters, for instance, which becomes the Planck length (1.6×10^{-35} m in SI, using $\hbar$ rather than h). Symbolic and numeric representations of the rest are left as an exercise for students.