

Dimensional Analysis

We have messed around a bit with mixing and matching units in the previous lecture in the context of electromagnetic units to come up with propagation speed and impedance of a coaxial cable, and also in the exercise of forming Planck units.

Now we turn to a more rigorous approach for solving problems dimensionally. This means that we should be able to get a functional solution up to some numerical constant. It's not a free ride: we have to supply a fair bit of intuition and limiting-case expectations. But it can get us out of logjams when we don't know how to approach problems at first.

Buckingham Pi Theorem

Named after Edgar Buckingham and presented in *Physical Review*, 4, 345 in 1914, the Pi theorem (named for a product, Π , rather than the stuff you eat) lays out a systematic way by which to establish appropriate scaling relations for physical problems.

The Central Idea

Units are arbitrary. Inches, millimeters, light years, all describe the same physical concept of length, with simple (some might say meaningless) conversion factors between them. Getting the physics right should not depend on unit systems. Any physical relation can be expressed as a series of terms, all moved to one side of the equation to produce zero on the right hand side:

$$Y_1 + Y_2 + Y_3 + \dots + Y_N = 0.$$

If we change some internal unit, like inches to centimeters, for instance, then we incur a factor of 2.54 to some rational power in each term, depending on how many powers of length appear in each term. In fact, since it is meaningless to add dimensionally inhomogeneous quantities, each Y_j term must have the same units and thus the same power of a particular unit like centimeters.

Then what's to keep us from making our physical relation dimensionless by dividing through by one of the Y_j ? For instance:

$$1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_1} + \dots + \frac{Y_N}{Y_1} = 0.$$

The simple power of this manipulation is that now one may change units at will without changing the equation, since the equation is now dimensionless. Inches, light years—who cares! More generally, we can cast any physical equation as

$$f(v_i) = 0, \tag{1}$$

which is a dimensionless function of n variables v_i , $i = 1, \dots, n$. Variables may include things like physical constants, important parameters in the problem, and—very importantly—a variable representing the actual quantity being sought (like a drag force, a tidal displacement, etc.).

The Buckingham Approach

Nothing profound yet, really. Buckingham suggested writing Eq. 1 in the following way:

$$f(\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_{n-r}) = 0, \tag{2}$$

where the n variables, v_i , employing r distinct fundamental units (length, time, mass, charge) have been re-cast into dimensionless “ Π ” variables of the form

$$\Pi_k = (v_1)^{\alpha_{k1}} (v_2)^{\alpha_{k2}} \dots (v_n)^{\alpha_{kn}}, \quad (3)$$

with the α_{ki} as rational powers. It is the fact that these new variables are *products* of all the others that earns the Π designation. Importantly, the α_{ki} powers are chosen so that each Π_k is dimensionless and thus invariant to changes of units (equivalently, scaling any of the r fundamental dimensions).

There are $n - r$ independent Π constructions, up to arbitrary powers of each Π_k . For instance, some function $f(\Pi_k)$ is not fundamentally distinct from a (different) function $g(\Pi_k^{\frac{8}{3}})$, achieving the same physical end.

We can usefully change the form of Eq. 2 to a different function of all but one of the Π_k , pulling out Π_1 (or any other) in the following way:

$$\Pi_1 = g(\Pi_2, \dots, \Pi_{n-r}, m), \quad (4)$$

where the index, m , is a subtle deal: an index to keep track of the possibility for multiple roots/solutions. One way to put this is that if we vary only Π_1 in Eq. 2 while keeping all other Π_k fixed, we must find *at least* one root (which is what allows us to adopt the form in Eq. 4 in the first place), but possibly more—in which case the index m is used to label multiple solutions. We will ultimately see an example of this for drag in viscous vs. inertial regimes.

That’s it. Really? Did we actually accomplish anything? Well, no. Just rearrangments after coercing our original problem into dimensionless variables via Eq. 3. If anything, the primary usefulness is recognizing that there are $n - r$ such combinations, and a formulaic way to write the equation. But we really need to see examples to make this come to life. And we’ll see that we still have to breathe physical intuition into the problem to make it puff up.

Example 1: Kepler’s Third Law

Starting gently on something we already know (and how to get at by comparing centripetal acceleration to gravity), let’s employ the method to arrive at Kepler’s third law. This is a relationship between period and orbital radius (semi-major axis). Hey! We already have two of the variables we need: T and a . We’ll expect it to depend on the mass of the central body, M , and also gravity, as embodied in the gravitational constant, G . Time to make a table!

i	v_i	units	notes
1	T	s	include variable for answer
2	a	m	obvious/sought dependency
3	M	kg	central body mass
4	G	$\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$	gravity, yeah? units via $GM/r^2 = \frac{\text{m}}{\text{s}^2}$

So $n = 4$ and $r = 3$ (s, m, kg), leaving only $n - r = 1$ unique combination of variables for a single Π value. Three of the units are simple, so we build them into something having the same units as G —namely, a^3/MT^2 . Divide G by this composite to get our single Π variable:

$$\Pi_1 = \frac{GMT^2}{a^3} \rightarrow f\left(\frac{GMT^2}{a^3}\right) = 0. \quad (5)$$

The Buckingham bus stops here. Now we have to apply physical intuition and limiting cases to make it real. Obviously, the function is more than just a multiple or power of Π_1 , or we end up with a trivial/nonsense case. Furthermore, there is no reason/call to apply any power at all. But we should apply an additive constant, K , to avoid the trivial solution:

$$\frac{GMT^2}{a^3} - K = 0,$$

which solves to

$$T^2 = \frac{Ka^3}{GM}.$$

We see the expected dependencies, and have recovered Kepler’s third law. In this case, K turns out to be rather large as numerical pre-factors go, at $K = 4\pi^2 \approx 40$. But the essential character is there, anyway.

Example 2: Tidal Bulge

What is the height of the tidal bulge on Earth due to the Moon’s pull? We already know the answer to be ~ 1 m. Feynman said: don’t ever attempt a problem unless you already know the answer. I suppose otherwise you will have no idea if you’re right. We will call the separation between Earth and Moon a , the bulge height Δh , and standard symbols elsewhere. Let’s jump straight to the table and build up as we go.

i	v_i	units	notes
1	Δh	m	sought quantity
2	a	m	Earth-Moon separation
3	R_\oplus	m	Earth radius: tide acts on extended body
4	G	$\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$	gravity!
5	M_m	kg	causing the fuss
6	g	m/s^2	what are we fighting against (or could do M_\oplus)

Now we have $n = 6$ and $r = 3$. We have three measures with units in meters, so we can quickly make Π_1 out of the sensible ratio

$$\Pi_1 = \frac{\Delta h}{R_\oplus}.$$

Let’s next knock out the G mess of units by constructing relevant parameters a , M_m , and g :

$$\Pi_2 = \frac{GM_m}{ga^2}.$$

At this point, we have used all six variables, but have one more to make before we reach $n - r$. Note that the two Π variables are disconnected from each other. This is a clear sign that we have an independent quantity/product to construct (that can’t be made from the two). We would like to relate the gravitational piece to R_\oplus . The easiest way to do this is $\Pi_3 = R_\oplus/a$. I was tempted to use physical intuition and make Π_3 almost identical to Π_2 except with $(a + R_\oplus)$ in place of a in the denominator—the rationale being that we expect the tidal phenomenon to relate to differential pull across the Earth’s extent. But this is inconsistent with the formal prescription of products, as in Eq. 3. (It still works out fine with this choice, FWIW, and even provides a factor of two getting closer to the truth.) So we end up with:

$$\frac{\Delta h}{R_\oplus} = f\left(\frac{GM_m}{ga^2}, \frac{R_\oplus}{a}\right).$$

Now for the physics intuition. We should expect the tidal influence to be linear in M_m , so the function is linear in Π_2 . We also expect a tidal response to look like the derivative of force (a differential effect), so a $1/r^3$ dependence. Thus the simplest “correct” function is the multiplication of Π_2 by Π_3 to get

$$\Delta h \sim \frac{GM_m R_\oplus^2}{ga^3} \approx \frac{\frac{1}{80} 4 \times 10^{14} 4 \times 10^{13}}{10 \cdot (4 \times 10^8)^3} = \frac{1}{320} \frac{4 \times 4 \times 4 \times 10^{27}}{4 \times 4 \times 4 \times 10^{25}} \approx \frac{1}{3} \text{ m},$$

where we have used the fact that the Moon’s mass is about 80 times less than Earth’s, and Earth’s GM product is very close to 4×10^{14} in SI units (something I just remember). So we certainly got into the right ballpark. Had we “violated” the rule of strictly product-based Π variable construction and used our $GM_m/g(a + R_\oplus)^2$ variable, we would have ended up a bit happier with 2/3 m. But the important thing is that we’re in the right ballpark and have all the scalings.

Had we used M_{\oplus} as a variable instead of g , we would have ended up with the rather symmetric (insightful?) construction:

$$\Delta h \sim R_{\oplus} \frac{M_m}{M_{\oplus}} \left(\frac{R_{\oplus}}{a} \right)^3 .$$

Had we done this, we would have seen that we would not need or want G , as getting rid of g in favor of M_{\oplus} leaves no other variables with a time dimension to play against G . We would have had $n = 5$, $r = 2$, and three Π variables essentially exactly following the simple/direct ratios in the expression above.

Synopsis

The Buckingham Pi theorem is not quite magic. It's just a way to organize a mess of relevant variables, helping us focus on physical reasoning. But in the end, it can't build the physics for us.