Water Waves

While we're dealing with fluids, we may as well cover waves on water (really any fluid). While waves are simple introductory material in most physics classes, water waves are not so simple, involving many different but relevant regimes. Let's dive in (chortle).

Buckingham Pi to Start

It's only natural that we reach for this friend to help us sort through the complexity. But what should we seek? We did a homework problem similar to this looking for wave speed. But we should strive to be more general. If we seek the *angular speed*, ω, instead of linear speed, we can get the *dispersion relation*, $\omega(k)$, which is the most valuable relation we can get when it comes to waves. So let's start a table:

i	v_i	units	notes
1	ω	1/s	sought
2	$ƛ$	m	$1/k$, or $\lambda/2\pi$
3	g	m/s^2	for big waves
4	γ	kg/s^2	for small waves
5	ρ	kg/m^3	relevant for small waves
6	d	m	depth of water

Even though we seek $\omega(k)$, I have more intuition for wavelength than for wavenumber. But at least the $ƛ = \lambda/2\pi$ variant will make for clean results (not missing annoying factors). Why is amplitude missing from the table? Much like the period and speed of a pendulum do not depend on amplitude (for small, linear-regime amplitudes), the same applies to waves. Think of waves as superpositions: a one-meter wave might be a superposition of two half-meter waves, but they had better travel at the same speed no matter how we mentally slice them up.

So we have $n = 6$ and $r = 3$ so that we expect three Pi variables. In the usual way, we build the first containing the thing we care about, the second as a simple length ratio, and the third to compare surface tension to gravity scales.

$$\Pi_1 = \frac{\omega^2 ƛ}{g},\ \Pi_2 = \frac{ƛ}{d},\ \Pi_3 = \frac{\gamma}{\rho g ƛ^2}.$$

Note the prevalence of the wavelength in all our variables. Putting this into the classic Buckingham form:

$$\omega^2 = \frac{g}{ƛ} f\left(\frac{ƛ}{d}, \frac{\gamma}{\rho g ƛ^2}\right). \tag{1}$$

We now have various physical regimes to consider. We could arrange these into a table organized by the two Pi variables in the function. The Π_2 variable sorts out deep from shallow, in relation to the wavelength. The Π_3 variable compares surface tension to gravity—but really the scale of the wavelength is the only free parameter in this comparison. So we might characterize associated waves as small or large.

	$\gamma \gg \rho g ƛ^2$ (small waves)	$\rho g ƛ^2 \gg \gamma$ (big waves)
$d \ll ƛ$	1. capillary, shallow	2. gravity, shallow
$d \gg ƛ$	3. capillary, deep	4. gravity, deep

The different regimes are numbered, and we will take them in turn—although not in numerical order.

#4: Large Deep Water Waves

In this scenario, we imagine the water waves have no idea how deep the ocean is: the physics does not reach that far. So the Π_2 variable is moot. We also suspect that surface tension has nothing to say about these large-scale waves. So we would not expect the resulting dispersion relation to contain γ. This is a very easy function: $f(\Pi_2, \Pi_3) = \Pi_2^0 \Pi_3^0 = 1$ (or some simple numerical factor). This leaves

$$\omega^2 = \frac{g}{\lambda\!\!\!^-} = gk.$$

It turns out that this is exactly right with no call for numerical factors.

Let's evaluate phase and group velocities:

$$v_{\rm ph} \equiv \omega/k = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}} = \sqrt{g\lambda\!\!\!^-} = \sqrt{\frac{g\lambda}{2\pi}};$$

$$v_{\rm gp} \equiv \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{g\lambda\!\!\!^-} = \frac{1}{2}v_{\rm ph}.$$

What does this mean? It means water is not a simple wave medium in which the form (even for a single narrow-band frequency) stays fixed and zooms through the water. When $v_{\rm ph} \neq v_{\rm gp}$, the medium is said to be *dispersive* (also describes condition of dependence of velocity on wavelength). Wave patterns do not survive intact. In this case, we would observe wave *crests* traveling at $v_{\rm ph}$, while the wave *amplitude envelope* (imagining a set of waves) travels only half as fast. This would appear as waves rolling through the set, coming into a wave packet, growing, then fading to zero amplitude as it outran the packet. The actual wave energy travels with the amplitude envelope, at the group velocity. Winds in the ocean at speed $v_{\rm wind}$ tend to amplify waves by pushing crests, so are most effective at creating waves with $v_{\rm ph} = v_{\rm wind}$, even though the energy created leaves at only half this speed.

An example for sanity: wave crests on the open ocean might be about $\lambda \approx 30$ m apart, meaning $\lambda\!\!\!^- \approx 5$ m, translating to $v_{\rm ph} \approx 7$ m/s and a group velocity of 3.5 m/s (and crest period of about 4 s).

#3: Deep Water Capillary Waves

We are still deep, so presumably Π_2 is still irrelevant. But we suspect surface tension should play a role, and that gravity should not. We can accomplish this with $f(\Pi_2, \Pi_3) = \Pi_3$, which gives us

$$\omega^2 = \frac{g}{\lambda\!\!\!^-}\frac{\gamma}{\rho g \lambda\!\!\!^{-2}} = \frac{\gamma}{\rho\lambda\!\!\!^{-3}} = \frac{\gamma k^3}{\rho}, \text{ or } \omega(k) = \sqrt{\frac{\gamma k^3}{\rho}}.$$

Again, this turns out to be exact with no need for scale factors. I think I'm gonna die! Before we get to the velocities, let's snoop out where these waves begin to be important. We will use the Π_3 variable as our guide, and when it crosses unity value. This happens when $\rho g \lambda\!\!\!^{-2} \approx \gamma$. This happens at

$$\lambda\!\!\!^- = \sqrt{\frac{\gamma}{\rho g}} \approx \sqrt{\frac{0.07}{10^3 \cdot 10}} \approx 2.6 \times 10^{-3}\,\text{m}, \text{ or } \lambda \approx 17\,\text{mm}.$$

So we're talking about relatively small waves. These are the things you see in advance of your finger dragged through water, like bunched-up ripples.

The velocities work out to:

$$v_{\rm ph} \equiv \frac{\omega}{k} = \sqrt{\frac{\gamma k}{\rho}} = \sqrt{\frac{\gamma}{\rho\lambda\!\!\!^-}} = \sqrt{\frac{2\pi\gamma}{\rho\lambda}}, \text{ and}$$

$$v_{\rm gp} = \frac{3}{2}\sqrt{\frac{\gamma k}{\rho}} = \frac{3}{2}v_{\rm ph}.$$

This time, the group velocity exceeds the phase velocity. For a wavelength around 6 mm, making $\lambda\!\!\!^- \approx 1$ mm, we get a phase velocity around $\sqrt{0.07} \approx 0.25$ m/s. Modest speed.

Combined Deep Water Waves

Notice something interesting in the behaviors of the velocities above. We see that for capillary waves, as the wavelength increases, the velocity decreases like $\lambda^{-\frac{1}{2}}$, while for large gravity waves the velocity increases like $\lambda^{\frac{1}{2}}$. As wavelength increases, we transition from capillary to gravity waves, so presumably there is a minimum velocity somewhere. The answer may be found by superimposing the dispersion relations to form a master relation for deep water:

$$\omega^2(k) = gk + \frac{\gamma}{\rho}k^3.$$

Now we have a phase velocity

$$v_{\text{ph}} = \sqrt{\frac{g}{k} + \frac{\gamma}{\rho}k} = \sqrt{g\lambdabar + \frac{\gamma}{\rho\lambdabar}}.$$

If we seek a minimum phase velocity as a function of wavelength (or equivalently $\lambdabar$), we set the derivative equal to zero:

$$\frac{\partial v_{\text{ph}}}{\partial \lambdabar} = \frac{1}{2}\left(g\lambdabar + \frac{\gamma}{\rho\lambdabar}\right)^{-\frac{1}{2}}\left(g - \frac{\gamma}{\rho\lambdabar^2}\right) = 0,$$

which happens when the last term is zero, or when

$$\lambdabar = \sqrt{\frac{\gamma}{\rho g}},$$

which is exactly the relation we evaluated for cross-over between gravity and capillary regimes, resulting in $\lambda = 16.6$ mm and $v_{\text{ph}} = 0.23$ m/s.

This means that any disturbance that travels **slower** than 0.23 m/s **will not** create waves of any sort (in water deeper than wavelength, or more than several centimeters deep). This is very useful for bugs who do not wish to be detected! I suggest you try it next time you are near a surface of still water.

The group velocity is a homely beast, and the algebra to ascertain its minimum is not worth committing to ink here. But it likewise has a minimum at $\lambdabar^2 = (3+\sqrt{12})\gamma/\rho g$, or $\lambdabar = 6.7$ mm, meaning $\lambda = 42.3$ mm, and $v_{\text{gp}} = 0.177$ m/s. When a rock drops into a still pond, waves travel away in an amplitude envelope, and the slowest part of the envelope travels at speed 0.177 m/s, leaving behind a still scene whose circular border expands at this speed.

For something pushing through the water (a duck, your finger, a boat), there will be an envelope pushing out in front, and the wavelength that has the same velocity as the object will bunch up into semi-static ripples/waves in front. Away from the minimum, there are two wavelengths that satisfy the relation: a gravity wave and a capillary wave. Look for the capillary ripples of characteristic wavelength bunched up in front of your finger rapidly pulled through water. The larger gravity wave is what sets the hull speed for a boat. As the boat gains speed, it builds up a larger-wavelength wave, and it ends up continuously trying to climb its own hill. Breaking out of this is called "planing," and is what speed boats do. Otherwise the longer the boat, the faster you can tolerate before the wavelength is comparable to your boat length.

#2: Shallow Gravity Waves

Recalling Eq. 1, we now deal with the situation where the depth is not negligible with respect to the wavelength. But we'll operate in the large-wave regime where gravity is dominant over surface tension. So we have

$$\omega^2 = \frac{g}{\lambdabar} f\left(\frac{\lambdabar}{d}\right).$$

So we need to consult our intuition. Do we think that waves speed up or slow down in shallower water? We might see interaction with the bottom as a frictional—or at least meddling—influence. We also may recall seeing waves on a beach overtake ones in front. Since velocity is proportional to ω, we think we want d in the denominator. So the function inverts Π_2 to yield

$$\omega^2 = \frac{g}{\lambda\!\!\!^{-}}\frac{d}{\lambda\!\!\!^{-}} = gdk^2.$$

Okay, so if $\omega(k) = \sqrt{gd}k$, then both the phase and group velocity are the same:

$$v_{\text{ph}} = v_{\text{gp}} = \sqrt{gd}.$$

This means that shallow waves are not dispersive. All wavelengths travel at the same speed, and any pattern (amplitude envelope) composed of any frequencies will travel in lock-step leaving the pattern intact. These waves really seem like "things," that persist and are not as ethereal as deep water waves.

Some examples: in a water depth of 2 m, $v \approx 4.5$ m/s, or about 10 m.p.h. In 1 m deep water, v becomes 3 m/s. If they arrive every 5–10 seconds, then we infer a wavelength $\lambda = 15$–30 m. These same waves in deep water have a phase velocity of 5–7 m/s and a group velocity that is half this.

Master Water Waves

All of the regimes previously covered can be codified in a single master dispersion equation:

$$\omega^2(k) = \left(gk + \frac{\gamma}{\rho}k^3\right)\tanh(dk) = \left(\frac{g}{\lambda\!\!\!^{-}} + \frac{\gamma}{\rho\lambda\!\!\!^{-3}}\right)\tanh\left(\frac{d}{\lambda\!\!\!^{-}}\right).$$

This form handles all cases and the transition regions between. Note that the hyperbolic tangent settles to 1.0 in deep water, becoming $d/\lambda\!\!\!^{-}$ in shallow water. We never treated case #1 in our table, but we can easily pull the result from here to find that $\omega^2 = \gamma dk^4/\rho$.

Wave Energy

We have focused on dispersion relations because these govern how waves behave, but we have not addressed wave energy. We start by noting that pressure is the same as energy density. That is, N/m^2 is the same as J/m^3. Water raised some height Δh carries an associated overburden pressure $\rho g\Delta h$. Thus the associated energy is $\rho g\Delta h$ times a volume, which is surface area times height, or $A\Delta h$. Energy per surface area is therefore something like $\rho g\Delta h^2$.

For waves, we say that the vertical displacement as a function of position is $\zeta(x, y)$, acting like our Δh above The energy in waves per unit area is

$$E_A = \rho g\left\langle\zeta^2\right\rangle,$$

where $\left\langle\zeta^2\right\rangle$ is a variance of the height averaged over space.

A sinusoidal wave with peak-to-peak amplitude α has a description $\zeta = \frac{\alpha}{2}\cos(\ldots)$, which means $\zeta^2 = \frac{\alpha^2}{4}\cos^2(\ldots)$, which has an average $\left\langle\zeta^2\right\rangle = \alpha^2/8$. So a water wave field with peak-to-peak amplitude of 1 m has $E_A \approx 10^3 \cdot 10 \cdot \frac{1}{8} = 1250\,\text{J/m}^2$. If this wave field is coming at us at 4 m/s, this constitutes 5000 W/m of intercepted energy. This is very close to the estimate in Lecture 3 for available wave power.

The energy, E_A, spreads out from the source, the area covered growing linearly in time as the radius and thus circumference grows linearly in time (at the group velocity). The thickness of the energy packet remains constant, for waves in a certain frequency domain. So absent dissipation, $E_A \propto t^{-1}$, which means that also $\left\langle\zeta^2\right\rangle \propto t^{-1}$, so that the scale of $\zeta \propto t^{-\frac{1}{2}}$. This is true individually for each frequency band, although the total areal energy density is diffused even faster since the breadth of the envelope broadens as longer wavelengths race ahead faster than the short wavelengths.

Example: High Dive

Let's say your friend goes off the high dive in cannonball form to make a big splash. At 70 kg, the initial energy is about $mgh \approx 2000$ J. If waves carry off half the energy (a total guess), or about 1 kJ, what is the characteristic wave amplitude a split second later when the mayhem is still confined to a 2 m radius around the entry point? Well, the area is $A = \pi R^2 \approx 10\,\mathrm{m}^2$, so $E_A \approx 100$ J/m^2, which is equal to $\rho g \left\langle \zeta^2 \right\rangle$. This makes $\left\langle \zeta^2 \right\rangle \approx 10^{-2}\,\mathrm{m}^2$. So the typical (RMS) vertical displacement is $\langle \zeta \rangle \approx 0.1$ m. This makes for a peak-to-peak of about $2\sqrt{2}\,\langle \zeta \rangle \approx 0.3$ m. This seems pretty plausible.