Sound

Propagation Speed

In a solid, picture a lattice of atoms bound in harmonic-like potentials (springs). Between adjacent atoms, mass m, is a spring with spring constant, k, and separation, a. The timescale for a compression to be felt/transmitted is $\tau \sim 1/\omega = \sqrt{m/k}$ (don't need the 2π because we don't need a full oscillatory cycle to transpire). The speed of propagation is then $c_{\rm s} = a/\tau = \sqrt{ka^2/m}$. We can play some games with this, dividing both the numerator and denominator within the radical by a^3. We are left with

$$c_{\rm s} = \sqrt{\frac{k/a}{m/a^3}} = \sqrt{\frac{E}{\rho}}.$$

The last, bold step acknowledges that a spring constant of a bar is the modulus of elasticity times the area divided by length: $k = E \cdot A/L$. For an individual atom in the lattice, the corresponding area is a^2 and the length of the spring is a. So $k = Ea$. The density is easy to see: mass per unit volume.

It's a simple and elegant formula. Let's try it out:

Material	E (Gpa)	ρ (kg/m^3)	$\sqrt{E/\rho}$ (m/s)	published $c_{\rm s}$ (m/s)
steel	200	8000	5000	6000
aluminum	70	2700	5000	5100
wood	10	1000	3000	3300–3600
diamond	1100	3500	17000	12000

So not too bad. How about water? The operative bonds here are hydrogen bonds, at $\varepsilon \approx 0.4$ eV. We saw previously that $E \sim \varepsilon/a^3$ and can easily enough compute $a = 0.3$ nm based on density and 18 grams per mole. The math gives $E \sim 2.5$ GPa (belly flop confirms it's hard), which then produces $c_{\rm s} \approx 1600$ m/s. The right answer is about 1500 m/s.

Air is far less dense, so the interactions come not via springs/bonds, but via collisions with thermal velocity streaming in between. Because the thermal velocity is not directed, having three directions to choose, we get a factor of $\sqrt{3}$ to deal with, and also the adiabatic constant γ, which is $\frac{7}{5}$ for diatomic air (related to the ration between changes in volume and corresponding changes in pressure). In the end, we have

$$c_s = v_{\rm th}\sqrt{\frac{\gamma}{3}} = \sqrt{\frac{3kT}{m}}\sqrt{\frac{\gamma}{3}} = \sqrt{\frac{\gamma kT}{m}}.$$

Plugging in $\gamma = 1.4$, $kT = \frac{1}{40}$ eV$\approx 4 \times 10^{-21}$ J, and $m = 29 \cdot 1.67 \times 10^{-27}$ kg, we get $c_{\rm s} \approx 350$ m/s. Really, sound temperature in air is just a function of $\sqrt{T}$ (in Kelvin). A table:

T (°C)	$c_{\rm s}$ (m/s)	condition
−50	298	at airliner altitude
−20	318	butt-cold
0	330	freezing
20	342	meh
40	353	hot

So we see a nearly 10% variation in the "normal" range of human experience. At altitude, the airliners see 300 m/s, so a typical Mach 0.8 flight is doing 240 m/s, or 540 m.p.h.

Incidentally, if we had associated the pressure of air as a sort of elastic modulus, $E \approx 10^5$ Pa, we would compute a sound speed $\sqrt{E/\rho} \approx 280$ m/s. Not outlandish. Throw in a $\sqrt{\gamma}$ and we're basically there! Indeed,

$c_s = \sqrt{\gamma p_0/\rho}$ is correct: using the ideal gas law $pV = NkT$, we can transform our sound speed expression $c_s = \sqrt{\gamma kT/m} = \sqrt{\gamma pV/Nm}$. Divide numerator and denominator by V, and recognize that N molecules of mass m apiece occupying a volume V constitutes a density for the gas, ρ: $c_s = \sqrt{\gamma p/(Nm/V)} = \sqrt{\gamma p/\rho}$, as above.

Sound Levels

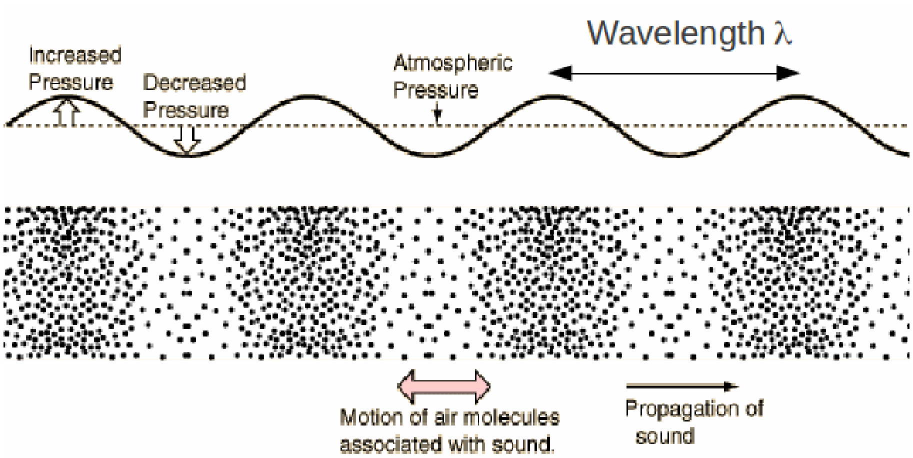
Effectively, we measure pressure/density fluctuations. We use a logarithmic scale, in decibels (dB), referenced to the fiducial power flux (intensity) of $I_0 = 10^{-12}$ W/m^2. This is the lowest conceivable (threshold) flux for human hearing. A whisper may be 30 times louder than this. We can then say

$$\mathcal{L} = 10 \log_{10}\left(\frac{I}{I_0}\right),$$

where I is the actual power per unit area, in W/m^2.

Pressure Terms

How does this relate to pressure? Think of a section of compressed air, taking, for instance, a length $\lambda/2$ along the longitudinal direction.



Now think of a "box" of air with cross-sectional area A and length $\lambda/2$ surrounding the compression. We think of the box as a spring. Actually, think of everything as a spring! We are going to compress the box of air by some Δx by applying a force on both ends of the box. The basic force is $F = p_0 A$, where p_0 is the baseline (nominal) pressure of the air (10^5 Pa, typically). In order to compress the box, we change the volume by $\Delta V = -A\Delta x$, and see a corresponding Δp. We do this quickly enough that no thermal energy comes into or goes out of our box (i.e., adiabatically), so that $pV^\gamma = p_0 V_0^\gamma =$ const. Putting all the volumes on the right side and pressures on the left, then differentiating, we find that $\Delta p/p_0 = -\gamma \Delta V/V_0$ (evaluating at $V = V_0$). Since $V_0 = A\lambda/2$, and $\Delta V = -A\Delta x$, we find that $\Delta p = 2p_0\gamma\Delta x/\lambda$, or as we'll use later, $\Delta x = \lambda\Delta p/2\gamma p_0$.

Getting back to forces, when we increase the force by Δp, we increase the force by $\Delta F = \Delta p A = k\Delta x$, relating to a spring with constant k. Thus we can make the association $k = A\Delta p/\Delta x = 2Ap_0\gamma/\lambda$. Why are we doing this? We want to relate pressure to power (or intensity), which means going via energy. And the energy in compressing a spring looks like $\Delta E = \frac{1}{2}k\Delta x^2$, which we compute to be

$$\Delta E = \frac{1}{2}k\Delta x^2 = \frac{1}{2}\frac{2Ap_0\gamma}{\lambda}\frac{\lambda^2\Delta p^2}{4\gamma^2 p_0^2} = \frac{\lambda}{4}\frac{A}{\gamma p_0}\Delta p^2.$$

Power is energy per time, and the time scale over which we compress our spring (as the sound wave moves through) is the time it takes to move through the length of the box, $\lambda/2$. So $\tau = \lambda/2c_s$, and power is

$$P = \frac{1}{2}\frac{c_s A}{\gamma p_0}\Delta p^2, \text{ so that } I = \frac{P}{A} = \frac{1}{2}\frac{c_s}{\gamma p_0}\Delta p^2 \rightarrow \Delta p_0 = \sqrt{\frac{I_0 \gamma p_0}{c_s}}.$$

Relating this to the reference power flux of $I_0 = 10^{-12}$ W/m^2, we calculate the corresponding scale for pressure fluctuations is 28 μPa. Actually, we overestimated since we assumed the entire box had the same overpressure. It's sinusoidal, so the average Δp^2 is half what we estimated, meaning we can get rid of the $\frac{1}{2}$ factor, as was done in the very last step above. So $\Delta p_0 = 20\,\mu$Pa. This means we can also put the sound level, in decibels, in pressure form:

$$\mathcal{L} = 20\log_{10}\left(\frac{\Delta p}{20\,\mu\text{Pa}}\right).$$

The 20 rather than 10 pre-factor owes to the square in the relation between Δp and I. We get 120 dB (nearly painful) when $\Delta p = 10^6\Delta p_0$, or 20 Pa. Think about this: when the fluctuations are only 0.02% of full-scale pressure (10^5 Pa), we can barely stand it. By the time we reach 140 dB (0.2%), we experience pain and may rupture an eardrum!

Example: An annoyingly loud motorcycle rumbles across a pedestrian bridge, 10 m above you (should not be there in the first place). You gauge the noise to be at the intolerable level of 120 dB. How much power goes into the sound generation?

120 dB means $\log_{10}(I/I_0) = 12$, or $I = 10^{12}I_0 = 1$ W/m^2. The area of a sphere 10 m in radius is about 1250 m^2. So the motorcycle must be pumping out 1250 W of acoustic energy. That's 1.7 horsepower, and about 5% of the total mechanical power available to a typical motorcycle.

Displacement

Finally, let's look at air molecule displacement. We found before that to squeeze our box, we related the squeeze amount to pressure via $\Delta x = \lambda\Delta p/2\gamma p_0$. At the extremes (edges of the box), air molecules move by half this amount (the ones in the center stay still, and the outer edges each come in by half the total squeeze amount). So we have displacement $\xi = \lambda\Delta p/4\gamma p_0$.

For example, for $f = 345$ Hz (a nice midrange characteristic of vocalization), $\lambda = 1$ m. At 120 dB, $\Delta p/p_0 = 2\times10^{-4}$, and $\xi \approx 36\,\mu$m. At 60 dB (conversational), it's 10^3 less, or 36 nm. At the threshold of human hearing, it's smaller still (by the same 10^3 factor: 0 dB), resulting in about a third of an Angstrom, less than atomic size. Assuming the displacement of the eardrum is comparable to the displacement of air molecules, then motion at the level of the thickness of a human hair is painful, while the detection threshold amounts to atomic-level displacement. And you thought LIGO was impressive, detecting waves at 0.001 times the proton dimension. Okay, that *is* impressive.

Helmholtz Resonator

Let's mess around with at least one acoustic generator, related to whistling, coke bottles, and other air-based resonant chambers. Picture a bulbous volume, V_0, with a neck (bottle-neck) of length ℓ and area A. The air inside the chamber is like a spring, and the air trapped in the neck is like a mass on the spring. There will be a resonant frequency!

We'll approach this via Newton's law, expressing the displacement of the bottle-neck mass as x, so that we have $m\ddot{x} = \Delta pA$ (force).$m = \rho A\ell$, and we know from our previous development that $\Delta p = -\gamma p_0\Delta V/V_0$. Moreover, if we push the bottle-neck mass into the bulbous volume a little bit, we find that $\Delta V = Ax$. So we can re-write our guiding equation as

$$\rho A\ell\ddot{x} = \Delta pA = \frac{\Delta p}{p_0}p_0A = -\gamma\frac{\Delta V}{V_0}p_0A = -\frac{\gamma p_0A^2}{V_0}x,$$

which we recognize as an oscillator with frequency

$$\omega^2 = \frac{\gamma p_0 A}{\rho \ell V_0} = c_{\rm s}^2 \frac{A}{\ell V_0},$$

so that the frequency is

$$f = \frac{\omega}{2\pi} = \frac{c_{\rm s}}{2\pi}\sqrt{\frac{A}{\ell V_0}}.$$

Let's try it out on a coke bottle. I'll say $V_0 = 0.5$ L, $\ell = 4$ cm, and the radius is 8 mm, making the area about 2×10^{-4} m^2. The ratio $A/\ell V_0$ works out to about 10, and the frequency is then about $1000/2\pi$, or in the neighborhood of 170 Hz.

Whistling forms a resonator in the mouth (higher-pitch requires the tongue to move up and make the resonator volume smaller). It is stimulated/pumped by alternating vortices related to the flow out of the mouth.

Sound Propagation in Temperature Gradient

We saw that sound speed varies as (only) the square root of absolute temperature. So what happens to sound in a medium sporting a temperature gradient (like our actual atmosphere)?

First, let's get a handle on relevant scales for the temperature gradient near the ground. In the daytime, with the sun overhead, the ground receives about 1000 W/m^2, absorbing maybe 80% of this. If we naively guess that half of this goes into heating the ground (conducts down) and half is radiated and convected to the surroundings, then we have 400 W into convection and radiation. Probably $h_{\rm conv} + h_{\rm rad} \sim 20$ W/m^2/K, since we're outside and have a cold blue sky. This would imply a ground surface temperature 20 C over the ambient air temperature. We can therefore imagine the air near the ground gets heated by 10 C compared to air maybe 100 m higher, for a temperature gradient of $\alpha \equiv dT/dz \sim -0.1$ K/m.

At night, the ground wants to equilibrate to the 255 K cold sky (effective blackbody temperature), but convection balances it out. So in power balance, $h_{\rm conv}\Delta T = \sigma(T^4 - T_{\rm sky}^4)$, where $T \sim 280$ K. This computes to 100 W/m^2, implying a $\Delta T \sim$ 10–20 C for $h_{\rm conv} \sim$ 5–10 in the still night. This is essentially the same as the daytime differential , so we may expect $\alpha \sim 0.1$ K/m again (same magnitude, different sign).

So what happens? Picture a plane wavefront traveling horizontally. The wave fronts extend vertically, probing different temperatures and therefore traveling at different speeds. Imagine two wheels on an axle rolling down the street, but one wheel has a little more friction than the other, so the thing bends in a gentle arc. So it goes with our wavefront. It will bend into an arc of some radius, R. After some time, it has bent through an angle θ. The arclengths differ. To picture it, imagine it's daytime, and the air is cooler as you go higher, so that sound travels slower the higher you go. The arc will bend upwards. If the radius, R, describes the bottom of the "beam" of sound, then its arclength is $R\theta$, while the upper, inner arc a height Δz above the bottom has radius $R - \Delta z$, traveling arclength $(R - \Delta z)\theta$. These paths are traversed in the same time, so

$$\frac{R\theta}{c_0} = \frac{(R - \Delta z)\theta}{c_z},$$

where c_0 denotes sound speed for the lower (reference) height, and c_z is the speed at relative height Δz. We can then form a ratio of speeds and relate to temperature:

$$\frac{c_z}{c_0} = \frac{R - \Delta z}{R} = 1 - \frac{\Delta z}{R} = \sqrt{\frac{T_z}{T_0}} = \sqrt{\frac{T_0 + \Delta T}{T_0}} = \sqrt{1 + \frac{\Delta T}{T_0}} \sim 1 + \frac{\Delta T}{2T_0}.$$

We can therefore relate $\Delta T/2T_0$ with $-\Delta z/R$. But we have defined the temperature gradient, $\alpha \equiv \Delta T/\Delta z = -2T_0/R$. Therefore when the scale of $\alpha \sim 0.1$, we find that $R = 6000$ m.

Acoustic paths tend to curve up in the daytime. This is why we don't tend to hear far away things very well in the daytime. After traversing some distance, or pathlength $x = R\theta$, the arc will curve up to some height,

h, such that $h = R\theta^2/2 = x^2/2R$, so that $x = \sqrt{2Rh}$. We therefore reach 100 m high (well above our heads) after traveling 1 km horizontally at this radius of 6 km.

At night, when a temperature inversion is more likely to exist, paths tend to curve downward. This is why we might hear distant trains at night.