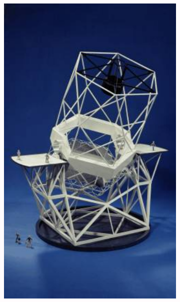


# Materials

Properties
Mechanics



## Why we need to know about materials

- Stuff is made of stuff
  - what should your part be made of?
  - what does it have to do?
  - how thick should you make it
- The properties we usually care about are:
  - stiffness
  - electrical conductivity
  - thermal conductivity
  - heat capacity
  - coefficient of thermal expansion
  - density
  - hardness, damage potential
  - machine-ability
  - surface condition
  - suitability for coating, plating, etc.





## Electrical Resistivity

- Expressed as $\rho$ in $\Omega \cdot m$
  - resistance = $\rho \cdot L/A$
    - where $L$ is length and $A$ is area
  - conductivity is $1/\rho$

| Material | $\rho$ ($\times 10^{-6}$ $\Omega \cdot m$) | comments |
|---|---|---|
| Silver | 0.0147 | $$ |
| Gold | 0.0219 | $$$$ |
| Copper | 0.0382 | cheapest good conductor |
| Aluminum | 0.047 | |
| Stainless Steel | 0.06–0.12 | |





## Thermal Conductivity

- Expressed as $\kappa$ in W $m^{-1}$ $K^{-1}$
  - power transmitted = $\kappa \cdot A \cdot \Delta T/t$,
    - where $A$ is area, $t$ is thickness, and $\Delta T$ is the temperature across the material

| Material | $\kappa$ (W $m^{-1}$ $K^{-1}$) | comments |
|---|---|---|
| Silver | 422 | room T metals feel cold |
| Copper | 391 | great for pulling away heat |
| Gold | 295 | |
| Aluminum | 205 | |
| Stainless Steel | 10–25 | why cookware uses S.S. |
| Glass, Concrete,Wood | 0.5–3 | buildings |
| Many Plastics | ~0.4 | room T plastics feel warm |
| G-10 fiberglass | 0.29 | strongest insulator choice |
| Stagnant Air | 0.024 | but usually moving… |
| Styrofoam | 0.01–0.03 | can be better than air! |

## Specific Heat (heat capacity)

- Expressed as $c_p$ in J $kg^{-1}$ $K^{-1}$
  - energy stored = $c_p \cdot m \cdot \Delta T$
    - where $m$ is mass and $\Delta T$ is the temperature change

| Material | $c_p$ (J $kg^{-1}$ $K^{-1}$) | comments |
|---|---|---|
| water | 4184 | powerhouse heat capacitor |
| alcohol (and most liquids) | 2500 | |
| wood, air, aluminum, plastic | 1000 | most things! |
| brass, copper, steel | 400 | |
| platinum | 130 | |





## Coefficient of Thermal Expansion

- Expressed as $\alpha = \delta L/L$ per degree K
  - length contraction = $\alpha \cdot \Delta T \cdot L$,
    - where $\Delta T$ is the temperature change, and $L$ is length of material

| Material | $\alpha$ ($\times 10^{-6}$ $K^{-1}$) | comments |
|---|---|---|
| Most Plastics | ~100 | |
| Aluminum | 24 | |
| Copper | 20 | |
| Steel | 15 | |
| G-10 Fiberglass | 9 | |
| Wood | 5 | |
| Normal Glass | 3–5 | |
| Invar (Nickel/Iron alloy) | 1.5 | best structural choice |
| Fused Silica Glass | 0.6 | |





## Density

- Expressed as $\rho = m/V$ in kg·$m^{-3}$

| Material | $\rho$ (kg $m^{-3}$) | comments |
|---|---|---|
| Platinum | 21452 | |
| Gold | 19320 | tell this to Indiana Jones |
| Lead | 11349 | |
| Copper, Brass, Steels | 7500–9200 | |
| Aluminum Alloys | 2700–2900 | |
| Glass | 2600 | glass and aluminum v. similar |
| G-10 Fiberglass | 1800 | |
| Water | 1000 | |
| Air at STP | 1.3 | |



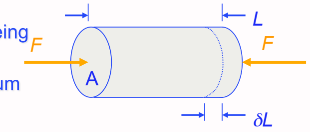


## Stress and Strain

- Everything is a spring!
  - nothing is *infinitely* rigid
- You know Hooke's Law:
  $F = k \cdot \delta L$
  - where $k$ is the spring constant (N/m), $\delta L$ is length change
  - for a given material, $k$ should be proportional to $A/L$
  - say $k = E \cdot A/L$, where $E$ is some elastic constant of the material
- Now divide by cross-sectional area
  $F/A = \sigma = k \cdot \delta L/A = E \cdot \varepsilon$ $\sigma = E \cdot \varepsilon$
  - where $\varepsilon$ is $\delta L/L$: the fractional change in length
- This is the stress-strain law for materials
  - $\sigma$ is the *stress*, and has units of pressure
  - $\varepsilon$ is the *strain*, and is unitless

## Stress and Strain, Illustrated

- A bar of material, with a force $F$ applied, will change its size by:
  $\delta L/L = \varepsilon = \sigma/E = F/AE$
- Strain is a very useful number, being dimensionless
- Example: Standing on an aluminum rod:
  - $E = 70\times10^{9}$ N·m$^{-2}$ (Pa)
  - say area is 1 cm$^2$ = 0.0001 m$^2$
  - say length is 1 m
  - weight is 700 N
  - $\sigma = 7\times10^{6}$ N/m$^2$
  - $\varepsilon = 10^{-4} \rightarrow \delta L = 100\ \mu$m
  - compression is width of human hair




$\sigma = F/A$

$\varepsilon = \delta L/L$

$\sigma = E\cdot\varepsilon$





## Elastic Modulus

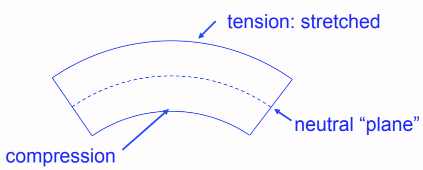
- Basically like a spring constant
  - for a hunk of material, $k = E(A/L)$, but $E$ is the only part of this that is intrinsic to the material: the rest is geometry
- Units are N/m$^2$, or a pressure (Pascals)

| Material | E (GPa) |
|---|---|
| Tungsten | 350 |
| Steel | 190–210 |
| Brass, Bronze, Copper | 100–120 |
| Aluminum | 70 |
| Glass | 50–80 |
| G-10 fiberglass | 16 |
| Wood | 6–15 |
| most plastics | 2–3 |





## Bending Beams



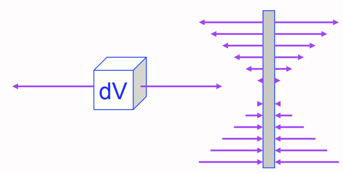

- A bent beam has a stretched outer surface, a compressed inner surface, and a neutral surface somewhere between
- If the neutral length is $L$, and neutral radius is $R$, then the strain at some distance, $y$, from the neutral surface is $(R + y)/R - 1$
  - $\varepsilon = y/R$
  - because arclength for same $\Delta\theta$ is proportional to radius
  - note $L = R\Delta\theta$
- So stress at $y$ is $\sigma = Ey/R$



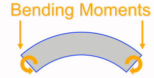


## In the Moment

- Since each mass/volume element is still, the net force is zero
  - Each unit pulls on its neighbor with same force its neighbor pulls on it, and on down the line
  - Thus there is no net moment (torque) on a mass element, and thus on the whole beam
    - otherwise it would rotate: angular momentum would change
  - But something is exerting the bending influence




Bending Moments



And we call this “something” the moment (balanced)

## What's it take to bend it?

- At each infinitesimal cross section in rod with coordinates ($x$, $y$) and area $dA = dxdy$:
  - $dF = \sigma dA = (Ey/R)dA$
  - where $y$ measures the distance from the neutral surface
  - the moment (torque) at the cross section is just $dM = y \cdot dF$
  - so $dM = Ey^2 dA/R$
  - integrating over cross section:

$$M = \int \frac{E}{R} y^2 dxdy = \frac{EI}{R}$$

  - where we have defined the "moment of inertia" as

$$I \equiv \int y^2 dxdy$$





## Energy in the bent beam

- We know the force on each volume element:
  - $dF = \sigma \cdot dA = E \cdot \varepsilon \cdot dA = (Ey/R)dA$
- We know that the length changes by $\delta L = \varepsilon dz = \sigma \cdot dz/E$
- So energy is: (z-direction)
  - $dW = dF \cdot \delta L = dF \cdot \varepsilon \cdot dz = E \cdot \varepsilon \cdot dA \times \varepsilon \cdot dz = E(y/R)^2 dxdydz$
- Integrate this throughout volume

$$W = \frac{E}{R^2} \int y^2 dxdydz = \frac{EIL}{R^2}$$

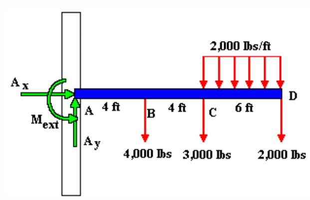
- So $W = M(L/R) \approx M\theta \propto \theta^2$
  - where $\theta$ is the angle through which the beam is bent





## Calculating beam deflection

- We start by making a free-body diagram so that all forces and torques are balanced
  - otherwise the beam would fly/rotate off in some direction

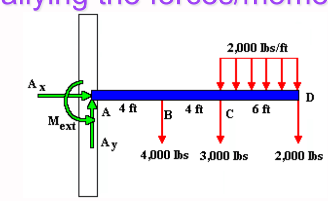



  - In this case, the wall exerts forces and moments on the beam (though $A_x$=0)
  - This example has three point masses and one distributed load





## Tallying the forces/moments

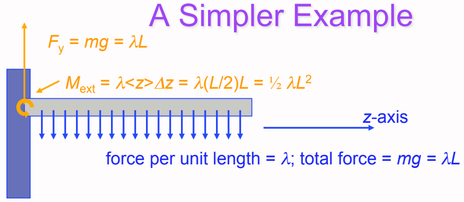



- $A_x = 0$; $A_y = 21{,}000$ lbs
- $M_{ext} = (4)(4000) + (8)(3000) + (14)(2000) + (11)(6)(2000) = 200{,}000$ ft-lbs
  - last term is integral:

$$M = \int_{x_1}^{x_2} \lambda x dx = \left[\lambda \frac{x^2}{2}\right]_{x_1}^{x_2} = \lambda \frac{x_1 + x_2}{2}(x_2 - x_1) = \lambda \langle x \rangle \Delta x$$

  - where $\lambda$ is the force per unit length (2000 lbs/ft)

## A Simpler Example




- A cantilever beam under its own weight (or a uniform weight)
  - $F_y$ and $M_{ext}$ have been defined above to establish force/moment balance
  - At any point, distance $z$ along the beam, we can sum the moments about this point and find:

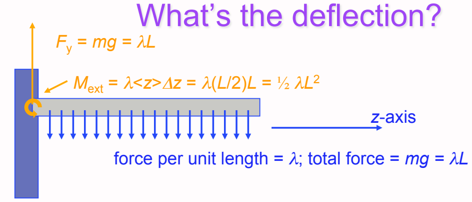
$$M_{\rm tot} = M_{\rm ext} - zF_y + \int_0^L \lambda(z - z')dz' = \frac{1}{2}\lambda L^2 - \lambda Lz + \lambda Lz - \frac{1}{2}\lambda L^2 = 0$$

  - validating that we have no net moment about any point, and thus the beam will not spin up on its own!





## What's the deflection?




- At any point, $z$, along the beam, the unsupported moment is given by:

$$M(z) = \int_z^L \lambda(z - z')dz' = \lambda\left[Lz - z^2 - \frac{L^2}{2} + \frac{z^2}{2}\right] = -\frac{mg}{2L}(z^2 - 2Lz + L^2)$$

- From before, we saw that moment and radius of curvature for the beam are related:
  - $M = EI/R$
- And the radius of a curve, $Y$, is the reciprocal of the second derivative:
  - $d^2Y/dz^2 = 1/R = M/EI$
  - so for this beam, $d^2Y/dz^2 = M/EI = -\frac{mg}{2EIL}(z^2 - 2Lz + L^2)$





## Calculating the curve

- If we want to know the deflection, $Y$, as a function of distance, $z$, along the beam, and have the second derivative…
- Integrate the second derivative twice:

$$\frac{d^2Y}{dz^2} = -\frac{mg}{2EIL}(z^2 - 2Lz + L^2) \rightarrow Y = -\frac{mg}{2EIL}\left(\frac{z^4}{12} - \frac{Lz^3}{3} + \frac{L^2z^2}{2} + Cz + D\right)$$

  - where $C$ and $D$ are constants of integration
  - at $z$=0, we define $Y$=0, and note the slope is zero, so $C$ and $D$ are likewise zero
  - so, the beam follows:

$$Y = -\frac{mg}{24EIL}(z^4 - 4Lz^3 + 6L^2z^2)$$

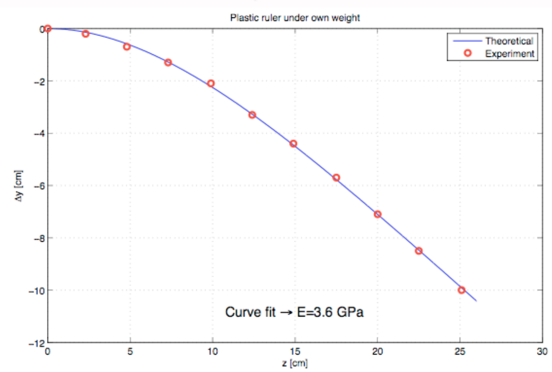
  - with maximum deflection at end: $Y_{\rm max} = \frac{mgL^3}{8EI}$



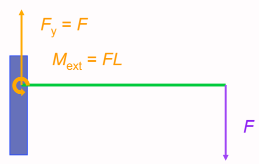


## Bending Curve, Illustrated

- Plastic ruler follows expected cantilever curve!



## End-loaded cantilever beam




- Playing the same game as before (integrate moment from *z* to *L*):

$$M(z) = (z - L)F \rightarrow \frac{d^2Y}{dz^2} = \frac{1}{R(z)} = \frac{M(z)}{EI} = \frac{F}{EI}(z - L)$$

  – which integrates to:

$$Y = \frac{F}{EI}\left(\frac{z^3}{6} - \frac{Lz^2}{2} + Cz + D\right)$$

  – and at $z$=0, $Y$=0 and slope=0 → $C = D = 0$, yielding:

$$Y = \frac{F}{6EI}(z^3 - 3Lz^2) \qquad Y_{\max} = \frac{FL^3}{3EI}$$





## Simply-supported beam under own weight


force per unit length = $\lambda$; total force = $mg = \lambda L$

- This support cannot exert a moment

$$M(z) = \int_z^L \lambda(z - z')dz' + \frac{1}{2}\lambda L(L - z) = \frac{1}{2}\lambda(Lz - z^2)$$

$$\frac{d^2Y}{dz^2} = \frac{\lambda}{2EI}(Lz - z^2) \rightarrow Y = \frac{\lambda}{2EI}\left(\frac{Lz^3}{6} - \frac{z^4}{12} + Cz + D\right)$$

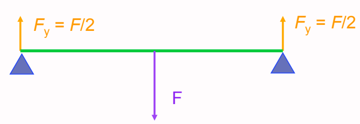
  – at $z$=0, $Y$=0 → $D$ = 0; at $z$=$L$/2, slope = 0 → $C = -L^3/12$

$$Y = \frac{mg}{24EIL}(2Lz^3 - z^4 - L^3 z) \qquad Y_{\max} = \frac{5}{384}\frac{mgL^3}{EI}$$





## Simply-supported beam with centered weight




- Working only from 0 < $z$ < $L$/2 (symmetric):

$$M(z) = F\left(z - \frac{L}{2}\right) + \frac{F}{2}(L - z) = \frac{Fz}{2} \rightarrow \frac{d^2Y}{dz^2} = \frac{Fz}{2EI}$$

  – integrating twice, setting $Y(0) = 0$, $Y'(L/2) = 0$:

$$Y = \frac{F}{12EI}(z^3 + Cz + D) \rightarrow Y = \frac{F}{12EI}\left(z^3 - \frac{3L^2 z}{4}\right)$$

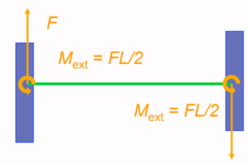
  – and the max deflection (at $z$=$L$/2):

$$Y_{\max} = \frac{FL^3}{48EI}$$





## S-flex beam




"walls" are held vertical; beam flexes in "S" shape

total M(z) = $2M_{ext} - Fz - F(L-z) = 0$ for all $z$

- Playing the same game as before (integrate moment from *z* to *L*):

$$M(z) = M_{\text{ext}} - F(L - z) = Fz - \frac{FL}{2} \rightarrow \frac{d^2Y}{dz^2} = \frac{1}{R(z)} = \frac{M(z)}{EI} = \frac{F}{2EI}(2z - L)$$

  – which integrates to:

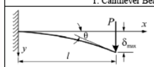
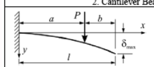
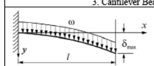
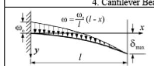
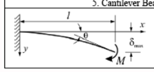
$$Y = \frac{F}{EI}\left(\frac{z^3}{6} - \frac{Lz^2}{4} + Cz + D\right)$$

  – and at $z$=0, $Y$=0 and slope=0 → $C = D = 0$, yielding:

$$Y = \frac{F}{2EI}\left(\frac{z^3}{3} - \frac{Lz^2}{2}\right) \qquad Y'(L) = 0 \qquad Y_{\max} = \frac{FL^3}{12EI}$$

as it should be

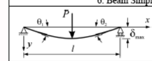
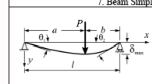
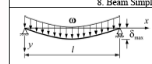
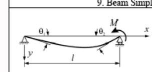
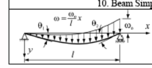
## Cantilevered beam formulae

| BEAM TYPE | SLOPE AT FREE END | DEFLECTION AT ANY SECTION IN TERMS OF $x$ | MAXIMUM DEFLECTION |
|---|---|---|---|
| 1. Cantilever Beam – Concentrated load $P$ at the free end | | | |
|  | $\theta = \frac{Pl^2}{2EI}$ | $y = \frac{Px^2}{6EI}(3l - x)$ | $\delta_{max} = \frac{Pl^3}{3EI}$ |
| 2. Cantilever Beam – Concentrated load $P$ at any point | | | |
|  | $\theta = \frac{Pa^2}{2EI}$ | $y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$<br>$y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$ | $\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$ |
| 3. Cantilever Beam – Uniformly distributed load $\omega$ (N/m) | | | |
|  | $\theta = \frac{\omega l^3}{6EI}$ | $y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$ | $\delta_{max} = \frac{\omega l^4}{8EI}$ |
| 4. Cantilever Beam – Uniformly varying load: Maximum intensity $\omega_o$ (N/m) | | | |
|  | $\theta = \frac{\omega_o l^3}{24EI}$ | $y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$ | $\delta_{max} = \frac{\omega_o l^4}{30EI}$ |
| 5. Cantilever Beam – Couple moment $M$ at the free end | | | |
|  | $\theta = \frac{Ml}{EI}$ | $y = \frac{Mx^2}{2EI}$ | $\delta_{max} = \frac{Ml^2}{2EI}$ |





## Simply Supported beam formulae

| BEAM TYPE | SLOPE AT ENDS | DEFLECTION AT ANY SECTION IN TERMS OF $x$ | MAXIMUM AND CENTER DEFLECTION |
|---|---|---|---|
| 6. Beam Simply Supported at Ends – Concentrated load $P$ at the center | | | |
|  | $\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$ | $y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$ | $\delta_{max} = \frac{Pl^3}{48EI}$ |
| 7. Beam Simply Supported at Ends – Concentrated load $P$ at any point | | | |
|  | $\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$<br>$\theta_2 = \frac{Pab(2l - b)}{6lEI}$ | $y = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2)$ for $0 < x < a$<br>$y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2 - b^2)x - x^3\right]$ for $a < x < l$ | $\delta_{max} = \frac{Pb(l^2 - b^2)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{(l^2 - b^2)/3}$<br>$\delta = \frac{Pb}{48EI}(3l^2 - 4b^2)$ at the center, if $a > b$ |
| 8. Beam Simply Supported at Ends – Uniformly distributed load $\omega$ (N/m) | | | |
|  | $\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$ | $y = \frac{\omega x}{24EI}(l^3 - 2lx^2 + x^3)$ | $\delta_{max} = \frac{5\omega l^4}{384EI}$ |
| 9. Beam Simply Supported at Ends – Couple moment $M$ at the right end | | | |
|  | $\theta_1 = \frac{Ml}{6EI}$<br>$\theta_2 = \frac{Ml}{3EI}$ | $y = \frac{Mlx}{6EI}\left(1 - \frac{x^2}{l^2}\right)$ | $\delta_{max} = \frac{Ml^2}{9\sqrt{3}\,EI}$ at $x = \frac{l}{\sqrt{3}}$<br>$\delta = \frac{Ml^2}{16EI}$ at the center |
| 10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity $\omega_o$ (N/m) | | | |
|  | $\theta_1 = \frac{7\omega_o l^3}{360EI}$<br>$\theta_2 = \frac{\omega_o l^3}{45EI}$ | $y = \frac{\omega_o x}{360lEI}(7l^4 - 10l^2x^2 + 3x^4)$ | $\delta_{max} = 0.00652\frac{\omega_o l^4}{EI}$ at $x = 0.519l$<br>$\delta = 0.00651\frac{\omega_o l^4}{EI}$ at the center |





## Lessons to be learned

- All deflections inversely proportional to $E$
  - the stiffer the spring, the less it bends
- All deflections inversely proportional to $I$
  - cross-sectional geometry counts
- All deflections proportional to applied force/weight
  - in linear regime: Hooke's law
- All deflections proportional to length cubed
  - pay the price for going long!
  - beware that if beam under own weight, $mg \propto L$ also (so $L^4$)
- Numerical prefactors of maximum deflection, $Y_{max}$, for same load/length were:
  - 1/3 for end-loaded cantilever
  - 1/8 for uniformly loaded cantilever
  - 1/48 for center-loaded simple beam
  - 5/384 ~ 1/77 for uniformly loaded simple beam
- Thus support at both ends helps: cantilevers suffer



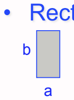


## Getting a feel for the *I*-thingy

- The "moment of inertia," or second moment came into play in every calculation

$$I \equiv \int y^2 dx dy$$

- Calculating this for a variety of simple cross sections:
- Rectangular beam:

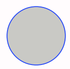



$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dy = a\left[\frac{y^3}{3}\right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{ab^3}{12} = \frac{A^2}{12}\frac{b}{a}$$

  - note the cube-power on $b$: twice as thick (in the direction of bending) is 8-times better!
  - For fixed area, win by fraction $b/a$

## Moments Later

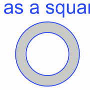
- Circular beam
  - work in polar coordinates, with $y = r\sin\theta$



radius, $R$

$$I = \int_0^R r dr \int_0^{2\pi} r^2 \sin^2\theta d\theta = \frac{\pi R^4}{4} = \frac{A^2}{4\pi}$$

  - note that the area-squared fraction ($1/4\pi$) is very close to that for a square beam (1/12 when $a = b$)
  - so for the same area, a circular cross section performs almost as well as a square
- Circular tube



inner radius $R_1$, outer radius $R_2$
or, outer radius $R$, thickness $t$

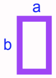
$$I = \int_{R_1}^{R_2} r dr \int_0^{2\pi} r^2 \sin^2\theta d\theta = \frac{\pi}{4}(R_2^4 - R_1^4) = \frac{\pi}{4}(R_2^2 + R_1^2)(R_2^2 - R_1^2) = \frac{A}{4}(R_1^2 + R_2^2)$$





## And more moments

- Circular tube, continued
  - if $R_2 = R$, $R_1 = R-t$, for small $t$: $I \approx (A^2/4\pi)(R/t)$
  - for same area, thinner wall stronger (until crumples/dents compromised integrity)
- Rectangular Tube




  - wall thickness = $t$

$$I = 2\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{\frac{b}{2}-t}^{\frac{b}{2}} y^2 dy + 2\int_{\frac{a}{2}-t}^{\frac{a}{2}} dx \int_{-\frac{b}{2}+t}^{\frac{b}{2}-t} y^2 dy = 2a\left[\frac{b^3}{24} - \frac{(\frac{b}{2}-t)^3}{3}\right] + 4t\frac{(\frac{b}{2}-t)^3}{3}$$

  - and if t is small compared to $a$ & $b$:

$$I \approx \frac{ab^2t}{2} + \frac{b^3t}{6}$$ and for a square geom.: $$I_{sq} \approx \frac{2a^3t}{3} \approx \frac{A^2}{24}\frac{a}{t}$$

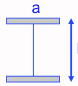
  - note that for $a = b$ (square), side walls only contribute 1/4 of the total moment of inertia: best to have more mass at larger $y$-value: this is what makes the integral bigger!





## The final moment

- The I-beam
  - we will ignore the minor contribution from the "web" connecting the two flanges




$$I = 2\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{\frac{b}{2}-t}^{\frac{b}{2}} y^2 dy = 2a\left[\frac{b^3}{24} - \frac{(\frac{b}{2}-t)^3}{3}\right] \approx \frac{ab^2t}{2}$$

  - note this is just the rectangular tube result without the side wall. If you want to put a web member in, it will add an extra $b^3t/12$, roughly
  - in terms of area = $2at$: $I \approx \frac{A^2}{8}\frac{b}{a}\frac{b}{t}$
- The I-beam puts as much material at high y-value as it can, where it maximally contributes to the beam stiffness
  - the web just serves to hold these flanges apart





## Lessons on moments

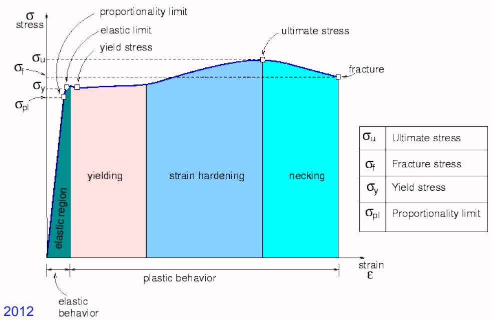
- Thickness in the direction of bending helps to the third power
  - always orient a 2×4 with the "4" side in the bending direction
- For their weight/area, tubes do better by putting material at high $y$-values
- I-beams maximize the moment for the same reason
- For square geometries, equal material area, and a thickness 1/20 of width (where appropriate), we get:
  - square solid: $I \approx A^2/12 \approx 0.083A^2$
  - circular solid: $I \approx A^2/4\pi \approx 0.080A^2$
  - square tube: $I \approx 20A^2/24 \approx 0.83A^2$
  - circular tube: $I \approx 10A^2/4\pi \approx 0.80A^2$
  - I-beam: $I \approx 20A^2/8 \approx 2.5A^2$

} 10× better than solid form (square tube, circular tube)
func. of assumed 1/20 ratio

- I-beam wins hands-down

## Beyond Elasticity

- Materials remain elastic for a while
  - returning to exact previous shape
- But ultimately plastic (permanent) deformation sets in
  - and without a great deal of extra effort








## Breaking Stuff

- Once out of the elastic region, permanent damage results
  - thus one wants to stay below the yield stress
  - yield strain = yield stress / elastic modulus

| Material | Yield Stress (MPa) | Yield Strain |
|---|---|---|
| Tungsten* | 1400 | 0.004 |
| Steel | 280–1600 | 0.0015–0.0075 |
| Brass, Bronze, Copper | 60–500 | 0.0005–0.0045 |
| Aluminum | 270–500 | 0.004–0.007 |
| Glass* | 70 | 0.001 |
| Wood | 30–60 | 0.0025–0.005 |
| most plastics* | 40–80 | 0.01–0.04 |

* ultimate stress quoted (see next slide for reason)





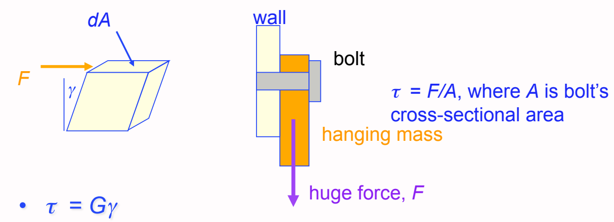
## Notes on Yield Stress

- The entries in red in the previous table represent ultimate stress rather than yield stress
  - these are materials that are brittle, experiencing no plastic deformation, or plastics, which do not have a well-defined elastic-to-plastic transition
- There is much variability depending on alloys
  - the yield stress for steels are
    - stainless: 280–700
    - machine: 340–700
    - high strength: 340–1000
    - tool: 520
    - spring: 400–1600 (want these to be elastic as long as possible)
  - aluminum alloys
    - 6061-T6: 270 (most commonly used in machine shops)
    - 7075-T6: 480





## Shear Stress




- $\tau = G\gamma$
  - $\tau$ is the shear stress ($N\cdot m^{-2}$) = force over area = $F/dA$
    - $dA$ is now the shear plane (see diagram)
  - $G$ is the shear modulus ($N\cdot m^{-2}$)
  - $\gamma$ is the angular deflection (radians)
- The shear modulus is related to $E$, the elastic modulus
  - $E/G = 2(1+\nu)$
  - $\nu$ is called Poisson's ratio, and is typically around 0.27–0.33

## Practical applications of stress/strain

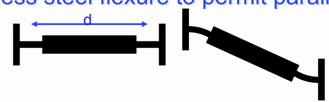
- Infrared spectrograph bending (flexure)
  - dewar whose inner shield is an aluminum tube 1/8 inch (3.2 mm) thick, 5 inch (127 mm) radius, and 1.5 m long
  - weight is 100 Newtons
  - loaded with optics throughout, so assume (extra) weight is 20 kg → 200 Newtons
  - If gravity loads sideways (when telescope is near horizon), what is maximum deflection, and what is maximum angle?
  - calculate $I \approx (A^2/4\pi)(R/t) = 2\times10^{-5}$ m$^4$
  - $E = 70\times10^9$
  - $Y_{max} = mgL^3/8EI$ = 90 μm deflection
  - $Y'_{max} = mgL^2/6EI$ = 80 μR angle
- Now the effect of these can be assessed in connection with the optical performance





## Applications, continued

- A stainless steel flexure to permit parallel displacement




  - each flexing member has length $L$ = 13 mm, width $a$ = 25 mm, and bending thickness $b$ = 2.5 mm, separated by $d$ = 150 mm
  - how much range of motion do we have?
  - stress greatest on skin (max tension/compression)
  - Max strain is $\varepsilon = \sigma_y/E$ = 280 MPa / 200 GPa = 0.0014
  - strain is $y/R$, so $b/2R$ = 0.0014 → $R = b/0.0028$ = 0.9 m
  - $\theta = L/R$ = 0.013/0.9 = 0.014 radians (about a degree)
  - so max displacement is about $d\cdot\theta$ = 2.1 mm
  - energy in bent member is $EIL/R^2$ = 0.1 J per member → 0.2 J total
  - $W = F\cdot d \rightarrow F$ = (0.2 J)/(0.002 m) = 100 N (~ 20 lb)





## Flexure Design

- Sometimes you need a design capable of flexing a certain amount without breaking, but want the thing to be as stiff as possible under this deflection
  - strategy:
    - work out deflection formula;
    - decide where maximum stress is (where moment, and therefore curvature, is greatest);
    - work out formula for maximum stress;
    - combine to get stress as function of displacement
    - invert to get geometry of beam as function of tolerable stress
  - example: end-loaded cantilever

$$Y_{\max} = \frac{FL^3}{3EI}$$

Δy is displacement from centerline (half-thickness)

$$M(z) = F(z-L) \rightarrow \max \text{ at } z = 0$$

$$\text{max strain, } \varepsilon = \frac{\Delta y}{R} = \frac{\Delta y M_{\max}}{EI} = \frac{FL\Delta y}{EI} \rightarrow \text{max stress, } \sigma_{\max} = E\varepsilon = \frac{FL\Delta y}{I}$$





## Flexure Design, cont.

- Note that the ratio $F/I$ appears in both the $Y_{max}$ and $\sigma_{max}$ formulae (can therefore eliminate)

$$\sigma_{\max} = \frac{F}{I}L\Delta y = \frac{3EY_{\max}}{L^3}L\Delta y = \frac{3EY_{\max}\Delta y}{L^2} = \frac{3EY_{\max}h}{2L^2}$$

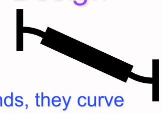
where $h = 2\Delta y$ is beam thickness

- If I can tolerate some fraction of the yield stress
  $\sigma_{max} = \sigma_y/\Phi$, where $\Phi$ is the safety factor (often chosen to be 2)

$$h = \frac{\sigma_{\max}}{E}\frac{2L^2}{3Y_{\max}} = \frac{\sigma_y}{\Phi E}\frac{2L^2}{3Y_{\max}} = \varepsilon_{\max}\frac{2L^2}{3Y_{\max}}$$

- so now we have the necessary (maximum) beam thickness that can tolerate a displacement $Y_{max}$ without exceeding the safety factor, $\Phi$
- You will need to go through a similar procedure to work out the thickness of a flexure that follows the S-bend type (prevalent in the Lab 2)

## Notes on Bent Member Flexure Design



- When the flex members have moments at both ends, they curve into more-or-less an arc of constant radius, accomplishing angle $\theta$
- $R = EI/M$, and $\theta = L/R = ML/EI$, where $L$ is the length of the flexing beam (not the whole assembly)
- $\sigma_{max} = E\varepsilon_{max} = E\Delta y/R = h\theta E/2L$, so $h = (\sigma_y/\Phi E)\times(2L/\theta)$
  - where $h = 2\Delta y$ and $R = L/\theta$





## Kinematic Design

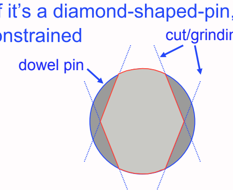
- Physicists care where things are
  - position and orientation of optics, detectors, etc. can really matter
- Much of the effort in the machine shop boils down to holding things where they need to be
  - and often allowing controlled adjustment around the nominal position
- Any rigid object has 6 degrees of freedom
  - three translational motions in 3-D space
  - three "Euler" angles of rotation
    - take the earth: need to know two coordinates in sky to which polar axis points, plus one rotation angle (time dependent) around this axis to nail its orientation
- Kinematic design seeks to provide minimal/critical constraint





## Basic Principles

- A three-legged stool will never rock
  - as opposed to 4-legged
  - each leg removes one degree of freedom, leaving 3
    - can move in two dimensions on planar floor, and can rotate about vertical axis
- A pin & hole constrain two translational degrees of freedom
- A second pin constrains rotation
  - though best if it's a diamond-shaped-pin, so that the device is not over-constrained

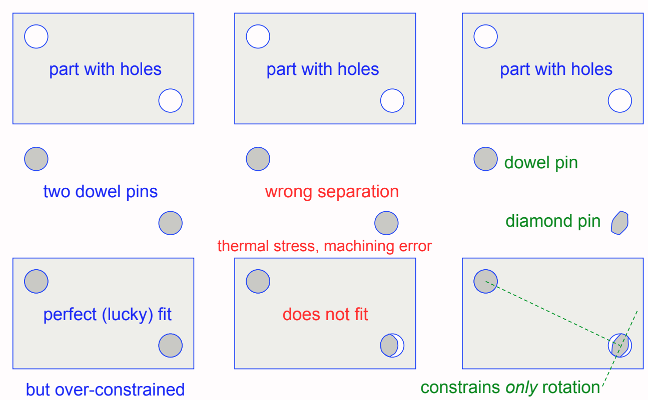



a diamond pin is a home-made modification to a dowel pin: sides are removed so that the pin effectively is a one-dim. constraint rather than 2-d





## Diamond Pin Idea




 diamond pin must be ground on grinder from dowel pin: cannot buy 

## Kinematic Summary

- Combining these techniques, a part that must be located precisely will:
  - sit on three legs or pads
  - be constrained within the plane by a dowel pin and a diamond pin
- Reflective optics will often sit on three pads
  - when making the baseplate, can leave three bumps in appropriate places
    - only have to be 0.010 high or so
  - use delrin-tipped (plastic) spring plungers to gently push mirror against pads





## References and Assignment

- For more on mechanics:
  - *Mechanics of Materials*, by Gere and Timoshenko
- For a boatload of stress/strain/deflection examples worked out:
  - *Roark's Formulas for Stress and Strain*
- Reading from text:
  - Section 1.5; 1.5.1 & 1.5.5; 1.6, 1.6.1, 1.6.5, 1.6.6 (3rd ed.)
  - Section 1.2.3; 1.6.1; 1.7 (1.7.1, 1.7.5, 1.7.6) (4th ed.)
- Additional reading on Phys239 website from 2010
  - http://www.physics.ucsd.edu/~tmurphy/phys239/lectures/twm_lecture6.pdf
  - very similar development to this lecture, with more text