Heat Transfer

Heat transfer is a story in three acts: conduction, convection, radiation. The encore, perhaps, is evaporative cooling. In many situations, one of the three primary mechanisms dominates, but there are also many in which all three are important, so we need the whole enchilada. We'll first take them in parts. We will also mostly confine our attention to equilibrium heat flow, rather than transient behavior, though we briefly visit transient timescales in our development of conduction.

To illustrate the separate parts and then the whole of heat transfer, we will consider an ice-fishing hut with a space heater inside to keep it toasty. The hut will be small, with a 1 meter square base, 2.5 meters high, and 2 cm-thick walls on all six sides (ignore the hole in the middle of the floor for fishing, which is not really so important, as mostly it's a place to go drink).

Conduction

Conduction describes heat transfer in solids, where the energy cannot be carried by mobile atoms, but must be carried either by lattice vibrations or free electrons (metals). For lattice vibration, the efficacy of conduction will depend on how ordered the lattice is, because this impacts the propagation of phonons (quantized lattice vibrations): a messy lattice promotes phonon scattering and thus more random-walk diffusion. In general, transient conduction indeed behaves as a diffusion process.

Estimating Thermal Conductivity

Can we order-of-magnitude our way into understanding conduction? First we need to define what we're after. If we imagine a plate of material with area A, thickness, s, and temperature difference, ΔT, presented across the plate, then we should expect the rate of heat flow in equilibrium to be $P_{\rm cond} = \kappa A \Delta T/s$, where κ is a new friend called thermal conductivity. Its units are W/m/K.

In a perfect crystal, if we ping an atom at the surface, the generated phonon(s) will race off to the other side, never to be heard from again. The energy injected will disappear into a black hole, as far as the hammer is concerned. In the presence of scattering, however, the phonon may come back to thermally interact with the external world on the side to which the heat is applied. In this case, the deeper one goes, the less likely one is going to see a phonon make it through the scattering gauntlet. So there will be a temperature gradient. Perhaps a better way to put it is that the material *can support* a temperature gradient, wherein scattering provides the support mechanism.

An Estimate by way of Timescales

How long will it take to boil an egg, or cook a turkey? We can put energy into the external surface at a rate limited by the conduction into the interior. The interior is initially at temperature T_0, and is distance R from the surface. So the initial power in is $P = \kappa A(T_{\rm h} - T_0)/R$. We need to heat the bulk of the material at its heat capacity, $c_{\rm p}$, requiring $\Delta E = \rho V c_{\rm p}(T_{\rm h} - T_0)$ of energy. The timescale is then set by $\tau \sim \Delta E/P = \rho V R c_{\rm p}/\kappa A$, which for a sphere is $\tau \sim \rho R^2 c_{\rm p}/3\kappa$. We can therefore define a thermal diffusion constant as $D_{\rm T} \sim 3\kappa/\rho c_{\rm p}$.

Recall that for a random walk process, the diffusion constant can be approximated as $D \sim \frac{1}{3}\lambda v$ (see Lecture 10). Making this association, we can estimate the thermal conductivity as $\kappa \approx \frac{1}{9}\lambda v \rho c_{\rm p}$. If we assume v is the phonon speed, and this is the sound speed, we'll get something on order $v \sim 2 \times 10^3$ m/s. For imperfect lattice arrangements (wood), we might expect phonon coherence only over a few lattice spacings, or a mean free path of $\lambda \sim 10^{-9}$ m. Meanwhile, the heat capacity for most things is $c_{\rm p} \approx 1000$ J/kg/K, and the density will be about 1000 kg/m^3. (Recall from Lecture 10 that $c_\mathrm{p} \sim 12000/A$ J/kg/K, where A is the atomic number of the substance in question.) Putting these numbers together produces $\kappa \sim 0.2$ W/m/K. Wood turns out to have thermal conductivities ranging from 0.05–0.4. As we get denser materials, or ones with better crystalline structure, this number will climb. But the heat capacity will go down somewhat when the atom occupying the lattice sites gets more massive, somewhat countering the density scaling. Applying this approach to air, we get about $\kappa \sim 0.01$ W/m/K, using a mean free path of about 100 nm and a "carrier" speed of 500 m/s (thermal velocity of air particle). The "real" answer is 0.024 W/m/K.

Let's look at some typical materials to get a feel for thermal conductivity:

Material	κ (W/m/K)	Comments
Stagnant Air	0.024	Only valid for still air; few mm or less
Styrofoam	0.03	mostly trapped air
wood	0.05–0.4	balsa to the densest stuff
plastics	0.2–0.5	also G-10 fiberglass at 0.3 (good strength)
stagnant water	0.6	for boundary layer (static, small scales)
concrete, brick, glass	~ 1	many building materials
rock	1–4	limestone to granite
stainless steel	15	why cookware uses this (don't burn self)
structural steel	~ 50	pure iron is 76
aluminum	250	floors should not be made of this!
silver	429	the king, with copper close behind, at 390

R-value of Building Materials

A closely related measure is the R-value of a material, signifying *resistance* to heat flow. High R values mean a higher degree of thermal insulation. R is defined in the unfortunate units of $^\circ\mathrm{F} \cdot \mathrm{ft}^2 \cdot \mathrm{hr} \cdot \mathrm{Btu}^{-1}$, and numerically evaluates to $\mathrm{R} \approx 5.7\frac{s}{\kappa}$, where s (thickness) and κ are expressed in the usual SI (MKS) units. Therefore a 10 cm thick wall of brick at $\kappa \approx 1$ W/m/K will have $\mathrm{R} \approx 0.6$. A typical exterior wall on a house will have $\mathrm{R} \sim 8$ if uninsulated, and $\mathrm{R} \sim 12$ if insulated. Note that R scales with thickness, and is therefore not an intrinsic material property, like κ. See `http://www.coloradoenergy.org/procorner/stuff/r-values.htm` for a compilation of R-values. Some of the values assume a convective limitation from an air boundary layer as well.

Back to the Ice Hut

Okay, if we make a hut furnished with $s = 2$ cm-thick wood walls with a thermal conductivity $\kappa = 0.3$, and we have a frigid -10°C outside and a toasty 20°C inside, we can immediately calculate the conducted power across the 12 m^2 of wall to be $P_\mathrm{cond} = \kappa A \Delta T/s = 0.3 \cdot 12 \cdot 30/0.02 = 5400$ W. It's going to take three space heaters from Target, which seems intuitively like too much.

Convection

Convection is a difficult process to compute accurately. Luckily, that's not our goal in this course. Convection boils down to conduction across the boundary layer, followed by the thermal energy being carried away by flowing fluid. We have seen that the thickness of the Blasius boundary layer (covered in Lecture 8) is $\delta \sim (x\nu/v)^{\frac{1}{2}}$, where x is the distance along the plate, ν the kinematic viscosity, and v the fluid velocity. For air flow at modest speeds of 1.5 m/s over something with characteristic dimension of 0.1 m, $\delta \sim$1 mm. The rate of heat conduction across this (static-air) boundary layer is $P_\mathrm{conv} \approx \kappa A \Delta T/\delta \equiv hA\Delta T$, where h is a convection constant in W/m^2/K. The foregoing value for δ yields $h \approx 0.024/10^{-3} = 24$ W/m^2/K.

In practice, air has $h \sim$1–2 W/m^2/K for self-induced airflow at modest (few degree) ΔT. For larger ΔT, a forced convection will apply (especially on vertical surfaces where hot air attains buoyancy), perhaps reaching

h ∼5–10. For light airs (outside), $h \sim 5$ W/m^2/K is a good choice. A breeze might get $h \sim 10$, and up from there as the flow increases. For vigorous flow, we might see h climb to 50 or higher. Water, having a lower kinematic viscosity, has a thinner boundary layer for the same flow, and because its static thermal conductivity is higher, the h values are almost 100 times higher for similar velocities. This is why water at the same temperature as air feels so much colder. But the drag forces are so much higher that we seldom find ourselves with very much velocity relative to the fluid, so the boundary layer thickens accordingly (albeit only as $v^{-\frac{1}{2}}$), and we do not get the full brunt of the factor-of-100 difference.

So back to the hut. What if we used thin, metal walls so that thermal conduction could be huge? How fast could convection pull away the heat? If the outside of our metal walls were at the same temperature as the inside (limiting case), and we use the most generally useful $h \approx 5$ W/m^2/K, our hut will be capable of emitting heat at a rate of $P_{\rm conv} = hA\Delta T \approx 5 \cdot 12 \cdot 30 = 1800$ W. This is three times less than the wooden-wall conduction would have said (and *far* smaller than the thin metal-walled hut would allow). So convection is a limiting factor.

Convection plus Conduction

The real hut has convection in series with conduction. The power in each channel must be equal in equilibrium—otherwise we build up or borrow thermal energy at the outside interface. We must solve for the temperature of the outer skin. If the inner wall skin is at $T_{\rm h}$, the outer skin at $T_{\rm skin}$, and the ambient air at $T_{\rm c}$, we have: $P_{\rm heater} = P_{\rm cond} = P_{\rm conv} = \kappa A(T_{\rm h} - T_{\rm skin})/s = hA(T_{\rm skin} - T_{\rm c})$. We can rearrange this to find $T_{\rm skin} = (\kappa T_{\rm h} + shT_{\rm c})/(\kappa + sh)$. For the foregoing values (recall $s = 2$ cm; $\kappa = 0.3$ W/m/K), we get $T_{\rm skin} \approx 12.5$ C, and find that the power is 1350 W. This is easily satisfied by a typical space-heater's capacity, which seems much more plausible. A more thorough job would also acknowledge convective coupling of the interior air to the interior wall, rather than just asserting the inside wall to be at T_h. But for the purposes of illustrating the basic ideas, it did not seem worth complicating the analysis and expressions in such a way.

A comparison of the values $\kappa/s \sim 15$ and $h \sim 5$ tell you that convection is the limiting process in the problem, but that both mechanisms deserve consideration. Note that κ/s is proportional to the inverse R-value (5.7/R, in fact), so h can be thought of this way as well. This means our choice of $h \sim 5$ W/m^2/K is roughly equivalent to R ∼ 1.

Radiation

This one is pretty familiar: $P_{\rm rad} = A\epsilon\sigma(T_{\rm h}^4 - T_{\rm c}^4)$, where $0 \leq \epsilon \leq 1$ is the emissivity, and σ is the Stefan-Boltzmann constant, $5.67 \times 10^{-8}\ {\rm W \cdot m^{-2} \cdot K^{-4}}$. For most things: humans, wood, plastics, rock, dirt, glass, paint—even white paint—$\epsilon > 0.8$, and may even safely be regarded as unity for approximate work. Only shiny things (metals) have low emissivity. Without working hard, aluminum has $\epsilon \approx 0.1$, though it can attain $\epsilon \approx 0.04$ when polished. A gold coating may be down to 0.01–0.02. But be careful, anodized aluminum can look shiny to the eye (clear anodizing), yet have a non-conductive surface that is opaque to infrared. Anodized aluminum falls in the $\epsilon > 0.8$ camp, even if shiny to the eye. Surface metal is where it's at.

The difference of T^4 values is somewhat inconvenient for solving together with conduction and convection, both of which are linear in ΔT. But if $T_{\rm h} \sim T_{\rm c}$ (in °K), then we may linearize: $P_{\rm rad} \approx 4A\epsilon\sigma T^3 \Delta T$, where T is some representative temperature in the problem—perhaps an average. Note that we now have a form similar to convection: $P_{\rm rad} = h_{\rm rad} A\Delta T$, where $h_{\rm rad} \equiv 4\epsilon\sigma T^3$. For typical situations: $\epsilon \sim 0.9$, $T \sim 290$ K, we get $h_{\rm rad} \approx 5.0$ W/m^2/K. So *it is very comparable to convection in magnitude.* This is a useful result. If a surface area is subject to convection and radiation to air and surrounding walls at the same ambient temperature, then you can generally get away with just doubling the usual $h_{\rm conv} \sim 5$ and killing two birds with one stone. On the other hand, if you take pains to eliminate one or the other (shiny surfaces or evacuated environment), you have only impacted half the heat flow. Cryogenic systems pursue both avenues aggressively—and use G-10 (or sometimes stainless steel) to stomp on thermal conduction without sacrificing mechanical strength.

The Whole Enchilada

Revisiting our ice-hut one last time, we consider all three thermal mechanisms at play. We additionally recognize that the walls inside couple to the air convectively (and mostly see each other radiatively, so no net flux there), so the inside walls have a skin temperature that is lower than the inside air. But the power flows from the air heater inside, through convection to the walls, through conduction to the outside skin, then convection *and* radiation from the outside skin to the environment. For the radiative piece, we will consider it to be a cold day with low clouds at the same temperature as the ambient air. If radiating to clear sky (ignoring the sun), the effective blackbody sky temperature would be $T_{\rm c} \approx 255$ K (greenhouse gases). Using a single value for $T_{\rm c}$, we have:

$$P_{\rm heater} = P_{\rm conv,in} = P_{\rm cond} = P_{\rm conv,out} + P_{\rm rad,out},$$

which breaks into:

$$P = h_{\rm in} A(T_{\rm h} - T_{\rm inner}) = \kappa A(T_{\rm inner} - T_{\rm outer})/s = h_{\rm out} A(T_{\rm outer} - T_{\rm c}) + h_{\rm rad} A(T_{\rm outer} - T_{\rm c}).$$

Note that the joint terms at the end share the same ΔT, that all terms scale with area, and that the radiative piece has used the linearized effective $h_{\rm rad} = 4\epsilon\sigma T^3$. After a little algebra,

$$T_{\rm outer} = \frac{h_{\rm in}\frac{\kappa}{s}T_{\rm h} + (h_{\rm out} + h_{\rm rad})(h_{\rm in} + \frac{\kappa}{s})T_{\rm c}}{h_{\rm in}\frac{\kappa}{s} + (h_{\rm in} + \frac{\kappa}{s})(h_{\rm out} + h_{\rm rad})},$$

from which it is easy to express $T_{\rm inner} = (h_{\rm in}T_{\rm h} + \frac{\kappa}{s}T_{\rm outer})/(h_{\rm in} + \frac{\kappa}{s})$. The power is also readily available given either of these two intermediate temperatures. To put numbers on it, our $T_{\rm h} = 20^\circ$C ice hut in ambient $T_{\rm c} = -10^\circ$C, with $s = 2$ cm walls of $\kappa = 0.3$ W/m^2/K wood will have $T_{\rm outer} \sim -3.4^\circ$C and $T_{\rm inner} = 0.5^\circ$C, requiring 700 W to maintain. This estimate uses $h_{\rm in} = 3$ W/m^2/K, $h_{\rm out} = 5$ W/m^2/K, $T = 266^\circ$K as an intermediate to $T_{\rm outer}$ and $T_{\rm c}$, yielding $h_{\rm rad} \sim 3.8$ W/m^2/K. Since $h_{\rm rad}$ scales as T^3, radiation will play less of a role as the temperature drops. Standing in the hut may still feel chilly, as the walls are cold and a person inside will feel radiative loss to them. Since we did not consider internal convection in our previous examples, having the inside walls at the inside air temperature, this is effectively the same as setting $h_{\rm in} \to \infty$. Doing so, we would end up with a power requirement of 2000 W. It makes sense that adding a radiation channel would increase the power required from the 1350 W we calculated for conduction and exterior convection alone.