Everyday Drag

Central in discussions of drag is *viscosity*. The first thing one should calculate in approaching a drag problem is the dimensionless *Reynold's Number*: $\mathrm{Re} \approx rv/\nu$; where r is the characteristic scale of the object, v is the velocity through the medium, and ν is the kinematic viscosity (more on this in a bit). There are two regimes for drag:

1. Viscous drag: $\mathrm{Re} \ll 1$; small things moving slowly
2. Inertial drag: $\mathrm{Re} > 100$, the drag is not a viscous phenomenon, but rather one of ram pressure

A crossover regime exists for Reynold's numbers of order 10. These regimes fall out nicely in a Buckingham Pi approach, which Prof. Diamond will cover later in this course.

A note on viscosity

The *kinematic viscosity*, ν, has units of [length]^2[time]$^{-1}$, while the *dynamic viscosity*, $\eta = \rho\nu$, has units of [mass][time]$^{-1}$[length]$^{-1}$ = [pressure][time]. The dynamic viscosity is perhaps more intuitive, in that water "should be" more viscous than air. Indeed, water has a dynamic viscosity of 10^{-3} Pa·s at a temperature of 20 C (down by a factor of 3 at boiling temperature), while air has a dynamic viscosity of $2 \times 10^{-5}\,\mathrm{Pa \cdot s}$ at 20 C (double this at 160 C). Meanwhile, the kinematic viscosity is often more useful (in Reynolds number, similar to diffusion constant), and for water is about 10^{-6} m^2/s, and air is about 1.5×10^{-5} m^2/s (larger than for water!). Here is a table of various fluid viscosities.

Substance	density ($\mathrm{kg \cdot m^{-3}}$)	$\eta\,(\mathrm{Pa \cdot s})$	$\nu\,(\mathrm{m^2 \cdot s^{-1}})$
air	1.3	2×10^{-5}	1.5×10^{-5}
water	1000	10^{-3}	10^{-6}
blood	1050	3×10^{-3}	3×10^{-6}
ethylene glycol	1100	1.6×10^{-2}	1.5×10^{-5}
olive oil	900	0.1	10^{-4}
corn syrup	1360	1.4	10^{-3}
peanut butter	1300	250	0.2

As Prof. Fuller will later show, the diffusion constant for a medium, $D \sim \frac{1}{3}\lambda v$, where the mean free path, $\lambda \approx 1/n\sigma$. For air at STP, we have 6×10^{23} particles in 22.4 ℓ, or 0.0224 m^3 for a number density of $2.7 \times 10^{25}\,\mathrm{m^{-3}}$ and a cross section of approximately $\pi r^2 \sim \pi(0.3\,\mathrm{nm})^2$, or $\sigma \approx 3 \times 10^{-19}$ m^2 for a mean free path of about 100 nm. The thermal velocity is $v = \sqrt{\frac{3kT}{2m}} \approx 350$ m/s (about the sound speed), so $D \sim 10^{-5}\,\mathrm{m^2 \cdot s^{-1}}$, strikingly similar to our value for ν in the table above.

Viscous Drag

How should viscous drag go? It should involve kinematic viscosity, ν, density of the medium, ρ, some scale of the object, r, and velocity, v. Putting these together dimensionally, one arrives at $F_\mathrm{d} \sim \rho\nu r v = \eta r v$.

Let's do a real example: what is the terminal velocity of a marble in corn syrup? The marble is about 1 cm in diameter, and we expect its speed to be in the neighborhood of 0.1 m/s. So the Reynold's number is about $\mathrm{Re} \approx (0.01\,\mathrm{m}) \cdot (0.1\,\mathrm{m/s})/(10^{-3}\,\mathrm{m^2/s}) = 1$. Really, $\mathrm{Re} < 100$ is laminar, and viscous-dominated, so the marble in corn syrup should be in the viscous regime. Therefore, the drag force will be $F_\mathrm{d} \sim \rho\nu r v = \eta r v$. When this equals mg of the marble, or $\frac{4}{3}\rho_\mathrm{m} g \pi r^3$, terminal velocity is achieved. So $v \sim 4\rho_\mathrm{m} g r^2/\eta$, evaluating to 1.4 m/s ($r = 5$ mm; $\rho_\mathrm{m} \sim 2 \times 10^3$). Seems fast. We should do an experiment.

Stokes drag, when done full-up, carries a factor of 6π along with the η. So we should divide our result by a factor of 20, to get 0.07 m/s. Not far from the initial guess, which honestly was *just* that, based on a mental picture of the (unperformed) experiment.

A dust grain in air, with diameter of about 100 μm (like a human hair diameter), will have a terminal velocity of around $v \approx \frac{1}{5}\frac{\rho_{\mathrm{obj}}}{\rho_{\mathrm{air}}} r^2 g/\nu$, or about 0.5 m/s if its density is about 1000 times that of air.

High Re Drag

When the speed increases, viscosity becomes less important, and ram pressure (inertial force) becomes the issue. In the frame of the object, the fluid rushes on, and must be displaced, stalled, or otherwise disrupted. The kinetic energy of the oncoming fluid is sapped in the process. If the cross-sectional area, A, intercepts a volume of fluid $V = Av\Delta t$ in time interval Δt, the kinetic energy "destroyed" per unit time (power) is $\frac{1}{2}\rho V v^2/\Delta t = \frac{1}{2}A\rho v^3$. The power to maintain this condition is equal to the drag force times the velocity, so that $F_{\mathrm{d}} \approx \frac{1}{2}\rho A v^2$.

Everyday Reynolds Numbers

The Reynolds number is Re $\sim rv/\nu$, with $\nu \approx 1.5 \times 10^{-5}$ m^2/s for air and $\nu \approx 10^{-6}$ m^2/s for water. Thus we have the following examples:

Action	r (m)	v (m/s)	ν (m^2/s)	Re
waving hand through air	0.1	5	1.5×10^{-5}	3×10^4
walking	0.5	2	1.5×10^{-5}	7×10^4
baseball pitch	0.05	40	1.5×10^{-5}	1.5×10^5
swimming	0.5	1	10^{-6}	5×10^5
car on freeway	1	30	1.5×10^{-5}	2×10^6
submarine at speed	4	10	10^{-6}	4×10^7
Boeing 747 at speed	4	300	1.5×10^{-5}	8×10^7

We do not personally experience viscous drag very often: only by watching tiny things in air/water do we tend to see this regime.

As a cute trick, if you forget the kinematic viscosity of water, imagine drawing a stick through the water and observing it vibrate, which happens at Re $\sim$ 2000. If the thin stick has diameter 4 mm (radius = 2 mm) and we observe the phenomenon at a moderately-brisk speed of 1 m/s, we have $\nu \sim rv/\mathrm{Re} \sim 2\times10^{-3}\cdot 1/2\times 10^3 = 10^{-6}$, right on target.

Drag Applications

We've seen, lived, and believed the scaling, also motivating it from a kinetic energy standpoint. We can lump the remaining ignorance into a dimensionless drag coefficient, c_{D}, of order unity. Then we have:

$$F_{\mathrm{drag}} = \frac{1}{2}c_{\mathrm{D}}\rho A v^2.$$

The coefficient of drag goes as follows (from the Wikipedia gods):

Object	c_{D}	comments
Boeing 747	0.03	uses chord-times wingspan
sphere	0.1–0.4	uses frontal area; depends on smoothness
best cars	0.25	frontal area
pickup truck	0.5	frontal area
tractor trailer	0.8	frontal area
man-bear-pig	1.0–1.4	most things fit here, if not built for streamline
brick	2.0	bad basketball shot

The range of c_D is not huge, within a factor of 2 of 0.5 for most things. Airplanes tend to use the chord of the airfoil times the wingspan (top area of wing) as the area, so the coefficient is not comparable directly to the others. Ships and submarines and swimming animals often use wetted area instead of frontal area, also lowering the number. For a trout, for instance, the wetted area $c_\mathrm{D} \sim 0.06$, while the frontal area $c_\mathrm{D} \sim 1.2$.

Gas Mileage

Let's consider the gas mileage for a pickup truck, with $c_\mathrm{D} \sim 0.5$, assuming all the energy goes into fighting drag. We use a frontal area of 4 m^2 (roughly a square of dimension 2 m), and a speed of 30 m/s to get a drag force of $0.5{\cdot}0.5 \cdot 1.3 \cdot 4 \cdot 900 \approx 1200$ N. The amount of work needed to go 1 mile (1.6 km) is then 1200 N$\times$1600 m, or about 2 MJ. Gasoline is about 10 kcal/g, so that 2 MJ (500 kcal) requires 50 g, or about 70 mℓ. But the combustion energy of the fuel is not delivered at 100% efficiency to the drive train. A typical efficiency would be 20% (about what you get from heat engine operating between 500 K and 350 K, realizing 50% of thermodynamic limit). So we need 0.35 ℓ to go one mile. Each liter will propel you about 3 miles, and with about 4 ℓ/gal, we get about 12 m.p.g. This is pretty close for a truck. Maybe too pessimistic, so 3 m^2 might be more realistic.

A car with half the drag coefficient and also half the projected frontal area will get four times the mileage, approaching 50 mpg.

Need a Lift?

Now let's augment our discussion of drag to consider the flight of an airplane. Let's not mess around with Bernoulli's principle or vortex voodoo. We'll just state that airplanes must fly by shoving air down. The downward force on the air is matched by the upward force on the plane, in level flight. Considering the vertical momentum of the air (zero before the plane comes along), we must impart $\Delta p = m_\mathrm{plane} g \Delta t$ of vertical momentum to the air in each time interval Δt. The volume of air involved is the "tube" with some effective cross-sectional area manipulated by the airfoil times the column length encountered in the time interval, so that the mass is: $m_\mathrm{air} = \rho A_\mathrm{eff} v \Delta t$, and this air acquires downward velocity v_down by the airfoil's passage. So the momentum imparted to the air per time interval Δt is $\rho A_\mathrm{eff} v v_\mathrm{down} = m_\mathrm{plane} g$.

Now the downward velocity should be related to the velocity of the plane in some sensible way. If we sit in the airfoil frame and draw the path of the airflow, we might draw the air deflecting moderately at an angle $\theta \sim 0.1$ rad, and $v_\mathrm{down} = v\theta$.

What is the effective area? It will certainly involve the wingspan, but what effective vertical height does the wing take hold of the surrounding air? We may naively guess this to scale like the wing chord (length along flow). Let's throw in some real numbers and see what pops out.

First, let's consider a Boeing 747 cruising at altitude at Mach 0.8 (typical), or around 240 m/s at altitude. How heavy is a B747? If we consider its giant tires, probably inflated to something like 45 p.s.i., or about 3 atm, or 3×10^5 Pa, then imagine how much footprint each makes on the ground, and multiply by the number of tires, we'll have an estimate. Guessing 32 tires (2 sets of 4$\times$4 trucks) and each with a footprint of (0.3 m)$\times$(0.6 m)$\sim$0.2 m^2 (these are person-size tires), we get a total area of 6 m^2 at 3×10^5 Pa, for a weight of 2×10^6 N, or 200 metric tons. Not comfortable with this? Remember the "Sully" plane floating in the Hudson, with people standing on the wings, still dry? The wings are at the bottom of the aircraft where they meet the fuselage. Let's say the water line comes up to 1/3 of the bottom-to-top fuselage span, displacing maybe a quarter of the cross-sectional area. If a B747 radius is 4 m and 40 rows at 1 m/row long, we have a displacement volume of about 500 m^3, or 500 tons of water. The geometric mean of the two estimates is 300 metric tons. (Wikipedia says max takeoff weight varies according to model from 330 to 440 tons.)

Using our estimate, we find the effective area to be $A_\mathrm{eff} = m_\mathrm{plane} g / \rho v^2 \theta$. At cruising altitude, the air density is significantly less than at sea level. The scale height of the atmosphere is 7 km, and the 747 cruises at maybe 1.5 scale heights, for a density about a quarter that of sea level, so 0.3 kg/m^3. Using $\theta = 0.1$, we get

$A_{\text{eff}} \sim 1600\ \text{m}^2$. If the wingspan is 60 m, then the vertical height is about 25 m! This is a big cross-section of the air. And this is the effective height of air shoved down at v_{down}. In reality, a much taller section of air is impacted by the passage of the plane, with diminishing response as a function of vertical distance.

How much power in the lift? In each time interval, Δt, we impart $\frac{1}{2}m_{\text{air}}v_{\text{down}}^2$ of kinetic energy to the air, resulting in a lift power of $P_{\text{lift}} = \frac{1}{2}(\rho A_{\text{eff}} v)(v\theta)^2 = \frac{1}{2}\rho A_{\text{eff}}\theta^2 v^3$, in which we can substitute our relation from before: $\rho A_{\text{eff}} v^2\theta = mg$ to get $P_{\text{lift}} = \frac{1}{2}mg\theta v$. This evaluates to 36 MW, requiring an engine force of $F = P/v$ of $\frac{1}{2}mg\theta \sim 1.5 \times 10^5$ N.

How about drag? I have two approaches for estimating the "drag area," Ac_{D}: one using the Wikipedia value for $c_{\text{D}} = 0.03$ corresponding to the wingspan times chord, or 60 m times maybe 5 m typical chord for $Ac_{\text{D}} \sim 10\,\text{m}^2$; or I can estimate the frontal area as 60 m times an average height of 1.5 m, and use a sports-car drag coefficient of about 0.25 to get $Ac_{\text{D}} \sim 25\,\text{m}^2$. If I take something in the middle, like 15 m^2, I get $F_{\text{d}} = \frac{1}{2}\rho Ac_{\text{D}}v^2 \sim 1.5 \times 10^5$ N, using $\rho = 0.3$ kg/m^3 and $v = 240$ m/s.

Is it a coincidence that the engine works as hard to push against drag as it does to produce lift? Not really. If we look at the original statement that $mg = \rho A v v_{\text{down}}$, we see that only the velocities are variable for a given airplane. If I cruise slower (less drag), v_{down} has to increase to compensate. We can say that $v^2\theta$ is constant, so that the engine force required to produce lift, $F = P_{\text{lift}}/v = \frac{1}{2}mg\theta$ scales as the inverse of the velocity squared. The sweet spot is achieved when the engine works as hard for lift as it does to fight drag. Since the goal of air travel is to go from point A to point B (rather than fly some fixed amount of time), the energy expended is proportional to force applied over the distance. So it is most important to seek the sweet spot in force rather than the sweet spot in power.

We could have established our value for θ based on this condition. It turns out that our guess was (honestly) lucky enough to do the job straight away.

As an aside, if we find a velocity, v_0, so that equal contributions are offered from a piece scaling like v^2 and a piece scaling like v^{-2}, we can write the force as $F = av^2 + b/v^2$, and achieve this equality of contributions at v_0 when $av_0^2 = b/v_0^2$, by which we can replace a with b/v_0^4. Doing so, we re-express $F = bv^2/v_0^4 + b/v^2$. Dividing out by b/v_0^2, we can say that force is proportional to $F \propto (v/v_0)^2 + (v_0/v)^2$. It is easy to show that this has an extremum (minimum) at $v = v_0$, justifying the statement that the sweet spot will have equivalent contributions from the two sources.

Glide Ratio

We can also get at the relation between lift and drag by realizing that a plane coasting under no power (engines dead) will descend at (we hope) a gentle angle. For a B747, this is something like a 20:1, or 0.05 radian descent. A Cessna might have a number around 10:1, and the space shuttle has (had) an alarming 2:1. Clearly a balance exists between lift and drag in this condition. We found above that the force required along the direction of travel was $\frac{1}{2}mg\theta$ to sustain lift. If we draw a force diagram with lift and drag perpendicular, we must tilt the pair forward in the absence of engine thrust to balance the downward gravity vector. The angle of tilt is $\theta/2$ if the drag force has a magnitude of $mg\theta/2$. At $\theta = 0.1$, this means a tilt of one part in 20. The drag vector opposes the direction of motion. If it gets tilted by a part in 20, it's because the glide path is tilted by this amount. So we again find consistency between our guess for θ and the glide slope for a B747.

We can also see this by looking at the perturbation to the velocity of the airflow. In horizontal flight, in the frame of the airplane, the wing ideally bends the velocity vector down by an angle θ, without changing its magnitude. The resultant vector between the original and new vector has a downward component of $\sim v\theta$, while the horizontal displacement is much smaller (and forward), at $v - v\cos\theta = v(1-\cos\theta) \approx \frac{1}{2}v\theta^2$. The ratio of horizontal to vertical components describes the angle of forward deflection of the airflow, which works out to $\theta/2$. If the airplane is in a steady gliding descent, the reacted air must be going straight down to offset the gravitational force, which must mean that the airplane has a downward glide slope of $\theta/2$.

Numbers for a Cessna

Did we just get lucky? Can a Cessna fly too? The smallest Cessna, a two-seater called the 152—whose characteristics I remember from once flying them—has a maximum takeoff weight of 1650 lbs (about 750 kg), and a wingspan of about 10 m. They cruise at about 110 m.p.h., or 50 m/s. They fly in dense sea-level air. The glide slope is 10:1, so we'll use $\theta = 0.2$. We get from this $A_{\text{eff}} \sim 12$ m^2, suggesting a vertical column equivalent of 1.2 m (proportionally small compared to B747, much owing to larger θ). The along-flight force required to sustain lift, $\frac{1}{2}mg\theta$, is 750 N. The value of Ac_{D} may be estimated two ways again: frontal wing area of 10 m times 0.2 m, plus another 1 m^2 for the cramped fuselage for a total area of 3 m^2 at a drag coefficient of 0.25, for $Ac_{\text{D}} \sim 0.75$ m^2. The wing chord is about 1.5 m, and the c_{D} will not be as good as for a 747: maybe 0.05. Together with the wingspan, this makes $Ac_{\text{D}} \sim 0.75$ m^2. Hey—the same! Now the force of drag at sea level and 50 m/s is 1200 N.

The Cessna 152 engine is rated at 110 horsepower = 82 kW. The thrust at 50 m/s is then 1640 N. This is pretty close to our sum of lift and drag along-flight forces required, and the two forces are also in the same ballpark. And I can tell you that the stated cruise speed is indeed running the engine at full power! Knowing this, the lift and drag should be about equal at 800 N, and our drag figure seems to be an overestimate.